

TROPICAL PLACTIC ALGEBRA, THE CLOAKTIC MONOID, AND SEMIGROUP REPRESENTATIONS

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ABSTRACT. A new tropical plactic algebra is introduced in which the Knuth relations are inferred from the underlying semiring arithmetics, encapsulating the ubiquitous plactic monoid \mathcal{P}_n . This algebra manifests a natural framework for accommodating representations of \mathcal{P}_n , or equivalently of Young tableaux, and its moderate coarsening — the cloaktic monoid \mathcal{K}_n and the co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$. The faithful linear representations of \mathcal{K}_n and ${}^{\text{co}}\mathcal{K}_n$ by tropical matrices, which constitute a tropical plactic algebra, are shown to provide linear representations of the plactic monoid. To this end the paper develops a special type of configuration tableaux, corresponding bijectively to semi-standard Young tableaux. These special tableaux allow a systematic encoding of combinatorial properties in numerical algebraic ways, including algorithmic benefits. The interplay between these algebraic-combinatorial structures establishes a profound machinery for exploring semigroup attributes, in particular satisfying of semigroup identities. This machinery is utilized here to prove that \mathcal{K}_n and ${}^{\text{co}}\mathcal{K}_n$ admit all the semigroup identities satisfied by $n \times n$ triangular tropical matrices, which holds also for \mathcal{P}_3 .

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INTRODUCTION

This paper introduces linear representations of the plactic monoid and its coarsening, called the cloaktic and the co-cloaktic monoids, together with a new encapsulating algebra. The **plactic monoid** $\mathcal{P}_I = \text{PLC}(\mathcal{A}_I)$ is the presented monoid $\mathcal{A}_I^* / \equiv_{\text{knu}}$. That is, the quotient of the free monoid \mathcal{A}_I^* over an ordered alphabet \mathcal{A}_I by the congruence \equiv_{knu} determined by the **Knuth relations** (also called **plactic relations**)

$$\begin{aligned} a c b &= c a b & \text{if } a \leq b < c, \\ b a c &= b c a & \text{if } a < b \leq c. \end{aligned} \tag{KNT}$$

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This monoid was first appeared in the context of Young tableaux (Knuth [26] and Schensted [45]) and has been followed by an extensive study of Lascoux and Schützenberger [30], establishing an important link between combinatorics and algebra via its bijective correspondence to semi-standard **Young tableaux** [25].

The combinatorics of \mathcal{P}_I is framed by Young tableaux and has been studied mainly in traditional perspective (e.g., Gröbner-Shirshov bases [4]). It has various applications (e.g., in symmetric functions [37], Kostka-Foulkes polynomials [31], and Schubert polynomials [32]). Related representations have looked at monoid algebras $K[\mathcal{P}_I]$ over a field K for a finite I [7, 27]. At the same time, Young tableaux has been found to be an important characteristic patterns in classical representation theory of groups [8, 10], especially in representations of the symmetric group [44] and algebraic combinatorics [36], as well as in combinatorics [35] and computer science [41]. Therefore, in this challenging arena, a direct algebraic description of the plactic monoid strengthens the mutual connections to classical representation theory, providing an additional algebraic-combinatorial approach to group representations.

To address this goal, this paper introduces

- a semiring structure in which the Knuth relations (KNT) are inferred from its arithmetics,
- meaningful coarser monoids of the plactic monoid, and
- linear representations of the plactic monoid and its coarsening.

These objects pave a new way to study the plactic monoid and Young tableaux systematically. To develop linear representation, we are assisted by several auxiliary objects (i.e., semigroups, tropical matrices, digraphs, and configuration tableaux), enhancing the interplay among them.

The fundamental structure of the monoids of our main interest, determined as a quotient by multiplicative congruences, makes their analysis rather difficult. To approach such monoids straightforwardly, we adopt the familiar concept of associating an algebra to a multiplicative group which allows one to study groups in terms of algebras, e.g., the correspondence between Lie groups and Lie algebras. This paper applies a similar concept to the plactic monoid \mathcal{P}_I by encapsulating \mathcal{P}_I in the **tropical plactic algebra** \mathbf{plc}_I — a new semiring structure — abbreviated **troplactic algebra**. The underlying structure of this algebra is a noncommutative idempotent semiring (Definition 2.4), generated by a set of ordered elements. In this algebra the Knuth relations (KNT) are intrinsically followed from the semiring arithmetics (Theorem 2.8).

An important property of \mathbf{plc}_I is that every product of its elements is uniquely decomposable to a sum of non-nested terms, each is a nondecreasing subsequence (Corollary 2.12). Furthermore, the generators of \mathbf{plc}_I admit the Frobenius, property (Lemma 2.15):

$$(\mathbf{a} + \mathbf{b})^m = \mathbf{a}^m + \mathbf{b}^m \quad \text{for any } m \in \mathbb{N}.$$

A dual version of the troplactic algebra, denoted \mathbf{plc}_I^\wedge , exists (Theorem 2.20) and is perfectly compatible with the co-mirroring map (CMR) below.

An immediate consequence of the equivalence $u \equiv_{\text{knu}} v$ of two words in \mathcal{P}_I is that the lengths $\overrightarrow{\text{len}}_{\mathcal{A}_I}(u)$ and $\overrightarrow{\text{len}}_{\mathcal{A}_I}(v)$ of longest nondecreasing subsequences (subwords) in u and v are the same. Furthermore,

$$\overrightarrow{\text{len}}_{\mathcal{A}_J}(u) = \overrightarrow{\text{len}}_{\mathcal{A}_J}(v) \quad \text{for every convex sub-alphabet } \mathcal{A}_J \text{ of } \mathcal{A}_I, \quad (\text{CLK})$$

which happens due to correspondence of \mathcal{P}_I with semi-standard Young tableaux (Proposition 6.9). The converse implication, however, does not hold in general, which leads to defining the **cloaktic monoid** $\mathcal{K}_I = \text{CLK}(\mathcal{A}_I)$ as the presented monoid $\mathcal{A}_I^* / \equiv_{\text{clk}}$, a moderate coarsen of \mathcal{P}_I . That is, $\mathcal{A}_I^* / \equiv_{\text{clk}}$ is the free monoid \mathcal{A}_I^* subject to the congruence \equiv_{clk} , determined by the relations (CLK). Then, the monoid homomorphism

$$\bar{\partial} : \mathcal{P}_I \longrightarrow \mathcal{K}_I$$

is surjective and preserves the congruence \equiv_{knu} .

As expected, the homomorphism $\bar{\partial}$ has a critical role in the construction of our linear representations of the plactic monoid, obtained from those of \mathcal{K}_I . Nevertheless, sometimes, \mathcal{K}_I appeared to be too coarser, which leads us to pursuing additional representable monoids that respect the congruence \equiv_{knu} . To receive such a monoid, we restrict the framework to finitely generated monoids \mathcal{A}_n^* and introduce the **co-mirroring** of letters

$$\text{co}\mathfrak{M}_n(a_\ell) := a_n a_{n-1} \cdots a_{n-\ell+2} a_{n-\ell} \cdots a_1, \quad a_\ell \in \mathcal{A}_n,$$

which induces the co-mirroring

$${}^{\text{co}}\mathfrak{M}_n(w) := {}^{\text{co}}\mathfrak{M}_n(a_{\ell_1}) \cdots {}^{\text{co}}\mathfrak{M}_n(a_{\ell_m}), \quad w = a_{\ell_1} \cdots a_{\ell_m}, \quad (\text{CMR})$$

over whole \mathcal{A}_n^* . It also determines the (lexicographic) order preserving surjective homomorphism

$$\partial_1 : \mathcal{A}_n^* \longrightarrow (\mathcal{A}_n^*)^{[1]} \subset \mathcal{A}_n^*, \quad w \longmapsto {}^{\text{co}}\mathfrak{M}_n(w).$$

The homomorphism ∂_1 extends inductively to a chain of endomorphisms $\partial_i : (\mathcal{A}_n^*)^{[i-1]} \longrightarrow (\mathcal{A}_n^*)^{[i]}$ of induced free monoids.

Our co-mirroring construction respects the congruence $\equiv_{\text{kn}} (Theorem 7.2)$. Thus, for the finitely generated plactic monoid \mathcal{P}_n , the monoid homomorphism

$$\partial_1 : \mathcal{P}_n \longrightarrow (\mathcal{P}_n)^{[1]} \subset \mathcal{P}_n$$

is injective — a monoid embedding. On the other hand, ∂_1 does not preserve the equivalence of \equiv_{clk} with the relations (CLK), but it determines the equivalence $\equiv_{\text{clk}}^{\text{co}}$, defined by

$$u \equiv_{\text{clk}}^{\text{co}} v \iff \partial_1(u) \equiv_{\text{clk}} \partial_1(v). \quad (\text{CCLK})$$

The **co-cloaktic monoid** ${}^{\text{co}}\mathcal{K}_n = {}^{\text{co}}\text{CLK}(\mathcal{A}_n)$ is constituted as ${}^{\text{co}}\mathcal{K}_n = \mathcal{A}_n^* / \equiv_{\text{clk}}^{\text{co}}$, that is accompanied with the surjective monoid homomorphism ${}^{\text{co}}\partial : \mathcal{P}_n \longrightarrow {}^{\text{co}}\mathcal{K}_n$, providing a second coarsening of \mathcal{P}_n .

The main goal of this paper is to construct linear representations of the plactic monoid \mathcal{P}_n in terms of tropical representations of the cloaktic monoid \mathcal{K}_n and the co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$. Therefore, a special focus is given to formulating the correspondences between \mathcal{P}_n to \mathcal{K}_n and ${}^{\text{co}}\mathcal{K}_n$ by relying upon Young tableaux.

We start by describing the combinatorial structure of the cloaktic monoid \mathcal{K}_n in terms of a tropalactic algebra \mathbf{plc}_I , obtained by using tropical matrices. Besides their conventional algebraic meanings, these matrices are also combinatorial entities, corresponding uniquely to weighted digraphs. As such, they intimately compose graph theory in tropical algebraic methodologies, exhibiting a useful interplay between algebra and combinatorics. The latter plays a major role in theoretical algebraic studies [2, 21] and in applications to combinatorics [3, 24], as well as in semigroup representations [15, 19] and automata theory [47, 48].

This mutual connection allows for producing a special class of tropical matrices that generate the finite tropical plactic algebra \mathfrak{A}_n (Theorem 4.2) along with recording lengths of the longest pathes in digraphs. In turn, these paths encode the longest nondecreasing sequences of the represented elements (Lemma 4.1). As being a tropalactic algebra, the multiplicative submonoid \mathfrak{A}_n^\times of \mathfrak{A}_n immediately admits the Knuth relations (KNT). More precisely, \mathfrak{A}_n^\times is isomorphic to the cloaktic monoid \mathcal{K}_n (Theorem 5.6). Thus, it introduces the faithful tropical linear representation

$$\mathfrak{U} : \mathcal{K}_n \xrightarrow{\sim} \mathfrak{A}_n^\times.$$

This isomorphism yields an efficient algorithm for computing the maximal lengths of all nondecreasing subwords of $w \in \mathcal{A}_n^+$ with respect to every convex sub-alphabet of \mathcal{A}_n (Algorithm 5.14). Similarly, for the co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$, we obtain the monoid isomorphism $\Omega : {}^{\text{co}}\mathcal{K}_n \xrightarrow{\sim} {}^\diamond \mathfrak{A}_n^\times$, whose image ${}^\diamond \mathfrak{A}_n$ is a dual tropalactic matrix algebra \mathbf{plc}_I^\diamond (Theorem 5.19). Both isomorphisms \mathfrak{U} and Ω are allocated with tropical characters that specify characteristic invariants.

An immediate result of the existence of the isomorphisms \mathfrak{U} and Ω is that both monoids \mathcal{K}_n and ${}^{\text{co}}\mathcal{K}_n$ admit all the semigroup identities satisfied by the monoid $\text{TMat}_n(\mathbb{T})$ of tropical triangular matrices (Corollaries 5.8 and 5.22). A particular form of these semigroup identities is constructed as

$$\Pi_{(C,p,q)} : \tilde{w} x \tilde{w} = \tilde{w} y \tilde{w}, \quad (\text{SID})$$

where $\tilde{w} := \tilde{w}_{(C,p,q)}$ is a fixed word over the variables $C = \{x, y\}$ that contains as factors all the possible words of length q over C in which no letter appears sequentially more than p times, and such that the words $\tilde{w} x \tilde{w}$ and $\tilde{w} y \tilde{w}$ also satisfy this law [14, 15]. With this form in place, $\text{TMat}_n(\mathbb{T})$ satisfies the identities (SID) with $p = q = n - 1$ by letting $x = uv$ and $y = vu$. The identities (SID) generalize the Adjan's identity of the bicyclic monoid [1], which is also faithfully represented by tropical matrices [19]. Also, $\text{TMat}_n(\mathbb{T})$ satisfies a recursive version of Adjan's identity [39].

With tropical representations of the monoids \mathcal{K}_n and ${}^{\text{co}}\mathcal{K}_n$ in hand, our next goal is to formulate the surjective monoid homomorphisms $\partial : \mathcal{P}_n \longrightarrow \mathcal{K}_n$ and ${}^{\text{co}}\partial : \mathcal{P}_n \longrightarrow {}^{\text{co}}\mathcal{K}_n$ explicitly, and to explore their properties as reflected in the representations $\mathfrak{U} : \mathcal{K}_n \xrightarrow{\sim} \mathfrak{A}_n^\times$ and $\Omega : {}^{\text{co}}\mathcal{K}_n \xrightarrow{\sim} {}^\diamond \mathfrak{A}_n^\times$. To this end, we

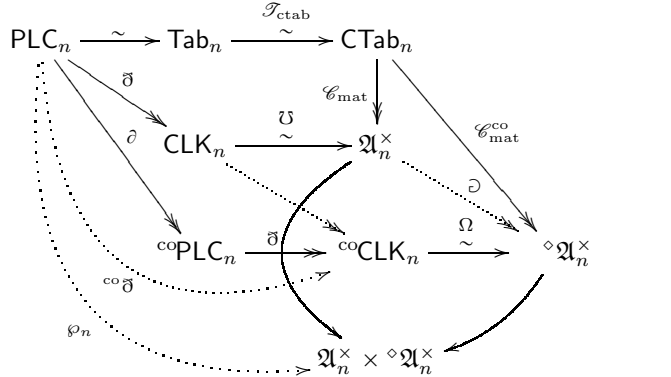
rely upon the well known one-to-one correspondence between the plactic monoid \mathcal{P}_n and semi-standard Young tableaux Tab_n (e.g., see [29]). In a sense, tableaux are graphical patterns that accommodate symbols, i.e., letters of the associated plactic monoid, with a deep combinatorial meaning. Nevertheless, an additional machinery is required to canonically frame Young tableaux by tropical matrices and to convert visual-combinatorial information to suitable numerical-algebraic data.

This converting machinery is provided by n -**configuration tableaux** CTab_n . That is, tableaux of fixed isosceles triangular shape that contain non-negative integers subject to certain structural laws, called configuration laws (Definition 6.11). Configuration tableaux correspond bijectively to semi-standard Young tableaux (Theorem 6.16) and are endowed with a self implementation of the Encoding Algorithm 6.19 that simulates the Bumping Algorithm 6.3 of Tab_n . Their fixed shape enables the introduction of a canonical reference system, employed to state their correspondence to tropical matrices in \mathfrak{A}_n^\times and ${}^\diamond\mathfrak{A}_n^\times$; thereby, to link tableaux with weighted digraphs. In this framework, encoding a letter $a_\ell \in \mathcal{A}_n$ in a configuration tableaux $\mathfrak{C}_w \in \text{CTab}_n$ is interpreted as multiplying the matrix $\mathcal{U}(w)$ by the matrix $\mathcal{U}(a_\ell)$ in \mathfrak{A}_n^\times or, dually, $\Omega(w)$ by $\Omega(a_\ell)$ in ${}^\diamond\mathfrak{A}_n^\times$. Accordingly, Tab_n and CTab_n are considered as multiplicative monoids whose operations are induced by letter encoding; hence, our tableau correspondences are realized as monoid homomorphisms.

With all desired components at our disposal, the tropical linear representation

$$\wp_n : \mathcal{P}_n \longrightarrow \mathfrak{A}_n^\times \times {}^\diamond\mathfrak{A}_n^\times$$

of the plactic monoid \mathcal{P}_n is obtained by composing the monoid homomorphisms of our main objects (Theorem 7.17), summarized by the diagram



This representation naturally induces a congruence on \mathcal{P}_n , and therefore also an equivalence relation on tableaux in Tab_n and CTab_n .

In the case of the plactic monoid of rank 3, \mathcal{P}_3 , the semigroup representation $\wp_3 : \mathcal{P}_3 \xrightarrow{\sim} \mathfrak{A}_3^\times \times {}^\diamond\mathfrak{A}_3^\times$ is faithful (Theorem 7.18), more precisely, it is an isomorphism. Consequently, we conclude that \mathcal{P}_3 admits all the semigroup identities satisfied by the monoid $\text{TMat}_3(\mathbb{T})$ of 3×3 triangular tropical matrices (Corollary 7.19); in particular, the identities $\Pi_{(C,2,2)}$ in (SID). Furthermore, relying upon the correspondence between reversal of words and transposition of tableaux, standard Young tableaux STab_n are shown to be faithfully realizable by tropical matrices (Theorem 7.20). As tableaux in STab_n bijectively correspond to elements of the symmetric group S_n , a tropical realization of S_n is obtained, linking the theory to Hecke algebras.

Tropical representation theory turns out to be applicable for studying characteristic properties of semigroups in places that the use of classical representation theory is limited. It provides an alternative approach for realization and for the exploration of algebraic-combinatorial objects; especially, their semigroup identities. These identities are of special interest in the theory of semigroup varieties [42], e.g., in finitely generated semigroups of polynomial growth [11, 46]. Applications of the ubiquitous plactic monoid in group representations are well known [10], e.g., in computing products of Schur functions in n variables which are the irreducible polynomial characters of the general linear group $\text{GL}_n(\mathbb{C})$, cf. [34]. Tropical linear representations pave the way to studying combinatorial aspects in classical representation theory, including a geometric perspective via projective modules [18, 38]. This paper ease the use of tropical representation by providing a solid bridge between tropical algebra and classical representation theory.

Paper outline.

For the reader convenience, this paper is designed as a self-contained document. In §1, we bring all the relevant definitions and properties of objects to be used in this paper, including basic examples. In §2, we introduce and study the new structure of tropical plactic algebra and its core object, called forward semigroup (Definition 2.2). A brief overview on tropical matrices and their associated digraphs opens §3, followed by the characterization of tropical corner matrices, providing faithful linear representations of forward semigroups. In §4 we employ corner matrices to construct an explicit troplactic algebra \mathfrak{A}_n and to establish the linkage between digraphs to nondecreasing subwords. The entire §5 is devoted to the introduction of the cloaktic monoid \mathcal{K}_n and the co-cloaktic monoid ${}^{\circ}\mathcal{K}_n$, including their linear representations by \mathfrak{A}_n and ${}^{\circ}\mathfrak{A}_n$, respectively. Young tableaux and configuration tableaux are discussed in §6, with a special emphasis on numerical functions that allow their mapping to matrices in \mathfrak{A}_n . Finally, in §7 we compose our various components to obtain representations and co-representations of configuration tableaux which eventually result as linear representations of the plactic monoid.

To help for a better understanding, our exposition involves many diagrams and examples, including of pathological cases.

1. PRELIMINARIES

To make this paper reasonably self contained, this section, except parts of §1.3, recalls the relevant notions and terminology, as well as definitions and properties of the algebraic structures to be used in the paper, starting with our special notations.

1.1. Notations.

Unless otherwise is specified, the capital letters I, J denote subsets of the natural numbers $\mathbb{N} := \{1, 2, \dots\}$, while \mathbb{N}_0 stands for $\{0, 1, 2, \dots\}$. The finite set $\{1, \dots, n\}$ is often denoted by N , for short. By “ordered” we always mean totally ordered.

Definition 1.1. *A subset J of an ordered set I is called **convex**, written $J \subseteq_{\text{cx}} I$, if for any $i, j \in J$, every $k \in I$ such that $i < k < j$, also belongs to J . We write $\{i : j\}$ for the convex subset of I determined by $i \leq j$.*

In other words, a convex subset is an “interval”, could be empty or a singleton. For a given n , we define $i' := n - i + 1$, for $i = 1, \dots, n$, ordered now reversely as $n' < (n-1)' < \dots < 1'$. Then $\{j' : i'\}$ has the same number of elements as $\{i : j\}$ has, and is again convex in N .

We write $L_m = [\ell_1, \ell_2, \dots, \ell_m]$ for a sequence of elements ℓ_t taken from $\{1, \dots, n\}$, where $t = 1, \dots, m$. This notation means that the indexing order of elements is preserved. We denote the sequence $[1, \dots, n]$ by $[1 : n]$, and write $[i : j]$ for the subsequence $[i, i+1, \dots, j]$ of $[1 : n]$.

- A subsequence S of L_m is notated as $S \subseteq L_m$, e.g., $[i : j] \subseteq [1 : n]$, for $1 \leq i \leq j \leq n$.
- $\text{len}(S)$ denotes the **length** of $S \subseteq L_m$, i.e., the number of its elements.
- S_k denotes a subsequence of length k , $k \geq 0$, where S_0 stands for the empty sequence.
- The notation $\prod_{s \in S_k} a_s$ stands for the product $a_{s_1} \cdot a_{s_2} \cdots a_{s_k}$, respecting the indexing of the sequence $S_k = [s_1, \dots, s_k]$.
- A subsequence $S_k = [s_1, s_2, \dots, s_k]$ of length k of L_m is
 - non-decreasing, denoted $S_k^{\uparrow} \subseteq L_m$, if $s_1 \leq s_2 \leq \dots \leq s_k$,
 - increasing if $s_1 < s_2 < \dots < s_k$,
 - non-increasing, denoted $S_k^{\downarrow} \subseteq L_m$, if $s_1 \geq s_2 \geq \dots \geq s_k$,
 - decreasing if $s_1 > s_2 > \dots > s_k$.
- S^{\uparrow} and S^{\downarrow} denote respectively non-decreasing and non-increasing subsequences of an arbitrary length.
- We denote the set of all non-decreasing subsets $S^{\uparrow} \subseteq L_m$ of L_m by $\text{Sq}^{\uparrow}(L_m)$.
- A non-decreasing subsequence $S^{\uparrow} \subseteq L_m$ is said to be **maximal** in L_m if L_m has no other non-decreasing subsequence $R^{\uparrow} \subseteq L_m$ such that $S^{\uparrow} \subsetneq R^{\uparrow}$. $\text{MSq}^{\uparrow}(L_m)$ denotes the subset of all maximal non-decreasing subsequences of L_m .
- $\text{Sq}^{\downarrow}(L_m)$ and $\text{MSq}^{\downarrow}(L_m)$ are defined similarly for non-increasing subsequences.

In this paper we deal only with finite sequences, which are also termed “words” in the context of semigroups.

1.2. Free monoids.

A **semigroup** $\mathcal{S} := (\mathcal{S}, \cdot)$ is a set of elements together with an associative binary operation. A **monoid** is a semigroup with identity element e . Any semigroup \mathcal{S} can be formally adjoined with an identity element e by declaring that $ea = ae = a$ for all $a \in \mathcal{S}$, so when dealing with multiplicative structures we work with monoids. We write a^i for $a \cdot a \cdots a$ with $a \in \mathcal{S}$ repeated i times, and formally identify a^0 with e , when \mathcal{S} is a monoid. An element o of \mathcal{S} is said to be **absorbing**, usually identified as $\mathbb{0}$, if $oa = ao = o$ for all $a \in \mathcal{S}$. \mathcal{S} is a **pointed semigroup** if it has an absorbing element o .

An Abelian semigroup \mathcal{S} is **cancellative** with respect to a subset $T \subseteq \mathcal{S}$ if $ac = bc$ implies $a = b$ whenever $a, b \in \mathcal{S}$ and $c \in T$. In this case, T is called a **cancellative subset** of \mathcal{S} , and it generates a subsemigroup in \mathcal{S} , also cancellative. Thus, we usually assume that T is a subsemigroup. A semigroup \mathcal{S} is **strictly cancellative** if \mathcal{S} is cancellative with respect to itself. The term “**congruence**” refers to an equivalence relation that respects the operation of its underlying semigroup carrier.

A **partially ordered semigroup** is a semigroup \mathcal{S} with a partial order \leq that respects the semigroup operation

$$a \leq b \quad \text{implies} \quad ca \leq cb, \quad ac \leq bc \quad (1.1)$$

for all elements $a, b, c \in \mathcal{S}$. A semigroup \mathcal{S} is **ordered** if the order \leq is a total order.

We recall some basic definitions from [14], for the reader’s convenience. As customarily, \mathcal{A}_I^* denotes the free monoid of finite sequences generated by a countably infinite totally ordered set $\mathcal{A}_I := \{a_\ell : \ell \in I\}$, called **alphabet**, of **letters** a_1, a_2, a_3, \dots . An element $w \in \mathcal{A}_I^*$ is called a **word** and, unless it is empty, is written uniquely as

$$w = a_{\ell_1}^{q_1} \cdots a_{\ell_m}^{q_m} \in \mathcal{A}_I^*, \quad \ell_t \in I, \quad q_t \in \mathbb{N}, \quad (1.2)$$

where $a_{\ell_t} \neq a_{\ell_{t+1}}$ for every t . We assume that the empty word, denoted e , belongs to \mathcal{A}_I^* , serving as the identity. As customarily, \mathcal{A}_I^+ stands for the free sub-semigroup obtained from \mathcal{A}_I^* by excluding the empty word e . When $|I| = n$ is finite, we write \mathcal{A}_n for the finite alphabet $\mathcal{A}_I = \{a_1, \dots, a_n\}$. A word is read from left to right, to distinguish this reading direction, when needed, we write \overrightarrow{w} for w .

The free monoid \mathcal{A}_I^* is endowed with the familiar **lexicographic order**, denoted $<_{\text{lx}}$, induced by the order of \mathcal{A}_I . A sub-alphabet \mathcal{A}_J of \mathcal{A}_I is said to be **convex**, written $\mathcal{A}_J \subseteq_{\text{cx}} \mathcal{A}_I$, if $J \subseteq_{\text{cx}} I$, i.e., J is a convex subset of I (Definition 1.1). We denote by $\mathcal{A}_{[i:j]}$ the convex sub-alphabet $\{a_i, \dots, a_j\}$ of $\mathcal{A}_n = \mathcal{A}_{[1:n]}$.

The **length** of a word $w \in \mathcal{A}_I^*$ of the form (1.2) is defined (by the standard summation) as $\text{len}(w) := \sum_{t=1}^m q_t$. Then, since a word w is a finite sequence of letters, $\text{len}(w)$ is well defined and finite, with $\text{len}(w)$ iff $w = e$ is the empty word e . A word $w \in \mathcal{A}_I^*$ is called **k -uniform** if each of its letters appears exactly k times. We say that w is **uniform** if w is k -uniform for some k .

A word $u \in \mathcal{A}_I^*$ is a **factor** of $w \in \mathcal{A}_I^*$, written $u|w$, if $w = v_1uv_2$ for some $v_1, v_2 \in \mathcal{A}_I^*$. When $w = v_1v_2$, the factors v_1 and v_2 are respectively called the **prefix** and **suffix** of w . If v_1 is of length k we say that v_i is the **k -prefix** of w , and denote it by $\text{pre}_k(w)$. Similarly, if $\text{len}(v_2) = k$, we say that v_i is the **k -suffix** of w , and denote it by $\text{suf}_k(w)$.

A word u is a **subword** of w , written $u \subseteq_{\text{wd}} w$, if w can be written as $w = v_0u_1v_1u_2v_2 \cdots u_mv_m$ where u_i and v_i are words (possibly empty) such that $u = u_1u_2 \cdots u_m$, i.e., the u_i are factors of u . Clearly, any factor of w is also a subword, but not conversely.

The following notion of n -power words was introduced in its full generality in [14, §3.1]. However, for the purpose of the present paper, it suffices to consider a restricted version of this notion, as described below. As seen later, these n -power words are the cornerstone of our systematic construction of semigroup identities.

Definition 1.2. Let $C := C_m$ be a finite (nonempty) alphabet, and let $n \in \mathbb{N}$. An **n -power word** $\tilde{w} := \tilde{w}_{(C,p,n)}$ is a nonempty word in C_m^+ such that:

- (a) Each letter $a_\ell \in C_m$ may appear in \tilde{w} at most p -times sequentially, i.e., $a_\ell^q \nmid \tilde{w}$ for any $q > p$ and $a_\ell \in C_m$;
- (b) Every word $u \in C_m^+$ of length n that satisfies rule (a) is a factor of \tilde{w} .

An n -power word is **uniform** if it is uniform as a word.

An n -power word needs not be unique in C_m^+ , and different n -power words may have different length. Often, n -power words $\tilde{w}_{(C,p,n)}$ can be concatenated to a new n -power word.

In the present paper, in the view of Theorem 1.8 below, we are mostly interested in the case of n -power words over the 2-letter alphabet $C_2 = \{x, y\}$. As we consider powerwords as generic words, we call the letters x, y **variables**.

Example 1.3. Suppose $C = \{x, y\}$.

- (i) $\tilde{w}_{(C,2,2)} = yx^2y^2x$ is a 2-power uniform word of length 6.
- (ii) $\tilde{w}_{(C,3,3)} = xy^3xyx^3y$ is a 3-power uniform word of length 10.

For more details on n -power words see [14, 15].

1.3. Co-mirroring and reversal.

Recall that the binary operation of a semigroup is associative.

Definition 1.4. A **semigroup homomorphism** is a map $\phi : \mathcal{S} \longrightarrow \mathcal{S}'$ that preserves the semigroup operation, i.e.,

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

for all a and b in \mathcal{S} . A **monoid homomorphism** is a semigroup homomorphism for which $\phi(e) = e'$. It is an **endomorphism** when $\mathcal{S} = \mathcal{S}'$.

A **presentation** of a monoid $\mathcal{M} = (\mathcal{M}, \cdot)$ (resp. semigroup) is a description of \mathcal{M} in terms of a set of generators $\mathcal{A}_I := \{a_\ell \mid \ell \in I\}$ and a set of relations $\Xi \subset \mathcal{A}_I^* \times \mathcal{A}_I^*$ on the free monoid \mathcal{A}_I^* (resp. on the free semigroup \mathcal{A}_I^+) generated by \mathcal{A}_I . The monoid \mathcal{M} is then presented as the quotient \mathcal{A}_I^*/Ξ of the free monoid \mathcal{A}_I^* by the set of relations Ξ , i.e., by a monoid homomorphism

$$\phi : \mathcal{M} \longrightarrow \mathcal{A}_I^*/\Xi, \quad \Xi \subset \mathcal{A}_I^* \times \mathcal{A}_I^*.$$

When $|I| = n$ is finite, we say that \mathcal{M} is **finitely presented**.

Definition 1.5. The **reversal** of a word $\vec{w} = a_{\ell_1}a_{\ell_2} \cdots a_{\ell_m}$ of length m in \mathcal{A}_I^* , $\ell_i \in I$, is the word $\overleftarrow{w} = a_{\ell_m}a_{\ell_{m-1}} \cdots a_{\ell_1}$ of the same length m .

The reversal \overleftarrow{w} of a word \vec{w} is therefore the rewriting of \vec{w} from right to left, while we formally set $\overleftarrow{e} := e$. It defines a bijective map

$$\text{rev} : \mathcal{A}_I^* \longrightarrow \mathcal{A}_I^*, \quad \vec{w} \longmapsto \overleftarrow{w},$$

which is not a monoid homomorphism, since $\text{rev}(uv) \neq \text{rev}(u)\text{rev}(v)$. However, $\text{rev}(uv) = \text{rev}(v)\text{rev}(u)$ for any $u, v \in \mathcal{A}_I^*$.

Restricting our ground alphabets to finite alphabets, we introduce the following operation:

Definition 1.6. The **co-mirror** of a letter a_ℓ in a finite alphabet \mathcal{A}_n is defined to be the word

$${}^{\text{co}}\mathfrak{M}_n(a_\ell) := a_n a_{n-1} \cdots a_{n-\ell+2} a_{n-\ell} \cdots a_1 = \prod_{\substack{t=n, \\ t \neq n-\ell+1}}^1 a_t. \quad (1.3)$$

(The co-mirror of the empty word e is formally set to be e .)

The **co-mirror** of a word $w = a_{\ell_1} \cdots a_{\ell_m}$ in \mathcal{A}_n^* is the word defined as

$${}^{\text{co}}\mathfrak{M}_n(w) := {}^{\text{co}}\mathfrak{M}_n(a_{\ell_1}) \cdots {}^{\text{co}}\mathfrak{M}_n(a_{\ell_m}), \quad (1.4)$$

written ${}^{\text{co}}\mathfrak{M}(w)$ when n is clear from the context.

Accordingly, for the concatenation $w = uv$ of two words u, v in \mathcal{A}_n^* we then have

$${}^{\text{co}}\mathfrak{M}_n(w) := {}^{\text{co}}\mathfrak{M}_n(u) {}^{\text{co}}\mathfrak{M}_n(v),$$

where $\text{len}({}^{\text{co}}\mathfrak{M}_n(w)) = (n-1)\text{len}(w)$ for any $w \neq e$. Thus, the co-mirroring of words determines a monoid endomorphism

$$\text{cmr} : \mathcal{A}_n^* \longrightarrow \mathcal{A}_n^*, \quad w \longmapsto {}^{\text{co}}\mathfrak{M}(w), \quad (1.5)$$

with $e \longmapsto e$.

Remark 1.7. The co-mirroring of a finite alphabet $\mathcal{A}_n = \{a_1, \dots, a_n\}$ introduces a submonoid in the free monoid \mathcal{A}_n^* , which we denote by $(\mathcal{A}_n^*)^{[1]}$, whose generators $a'_\ell = {}^{\text{co}}\mathfrak{M}_n(a_\ell)$ are again (totally) ordered as $a'_1 < \dots < a'_n$. In fact, as can be seen from (1.3), this order is just the lexicographic order $<_{\text{lx}}$ of \mathcal{A}_n^* , and thus is compatible with the initial order of \mathcal{A}_n .

Applying the co-mirroring map (1.5) inductively, we have a chain of submonoids, a filtration,

$$\mathcal{A}_n^* \supset (\mathcal{A}_n^*)^{[1]} \supset (\mathcal{A}_n^*)^{[2]} \supset (\mathcal{A}_n^*)^{[3]} \supset \dots,$$

together with the surjective homomorphisms

$$\mathcal{A}_n^* \xrightarrow{\partial_1} (\mathcal{A}_n^*)^{[1]} \xrightarrow{\partial_2} (\mathcal{A}_n^*)^{[2]} \xrightarrow{\partial_3} (\mathcal{A}_n^*)^{[3]} \xrightarrow{\partial_4} \dots$$

Each surjection $\partial_i : (\mathcal{A}_n^*)^{[i-1]} \rightarrow (\mathcal{A}_n^*)^{[i]}$ is also injective, preserving the lexicographic order of \mathcal{A}_n^* , and thus it is an isomorphism.

Note that ∂_i does not necessarily preserve relations nor presentations over \mathcal{A}_n^* .

1.4. Semigroup identities.

A (nontrivial) **semigroup identity** is a formal equality of the form

$$\Pi : u = v, \tag{1.6}$$

where u and v are two different (nonempty) words in the free semigroup \mathcal{A}_I^+ , cf. Form (1.2). For a monoid identity, u and v are allowed to be the empty word as well. Therefore, a semigroup identity Π determines a single relation $\Pi \in \mathcal{A}_I^+ \times \mathcal{A}_I^+$ on the free semigroup \mathcal{A}_I^+ .

We say that an identity $\Pi : u = v$ is an **n -variable identity** if u and v involve together exactly n letters of \mathcal{A}_I . An identity Π is said to be **balanced** if the number of occurrences of each letter $a_i \in \mathcal{A}_I$ is the same in u and in v . Π is called **uniformly balanced** if furthermore the words u and v are k -uniform for some k . The **length** $\ell(\Pi)$ of Π is defined to be $\ell(\Pi) := \max\{\ell(u), \ell(v)\}$, clearly $\ell(u) = \ell(v)$, when Π is balanced.

A semigroup $\mathcal{S} := (\mathcal{S}, \cdot)$ satisfies the semigroup identity (1.6) if

$$\phi(u) = \phi(v) \quad \text{for every homomorphism } \phi : \mathcal{A}_I^+ \rightarrow \mathcal{S}.$$

Concerning the existence of identities it suffices to consider 2-variable identities:

Theorem 1.8 ([14, Theorem 3.10]). *A semigroup $\mathcal{S} := (\mathcal{S}, \cdot)$ that satisfies an n -variable identity $\Pi : u = v$, for $n \geq 2$, also satisfies a refined 2-variable identity $\hat{\Pi} : \hat{u} = \hat{v}$ of exponents $\{1, 2\}$, i.e., each letter in \hat{u} and in \hat{v} appears sequentially at most twice.*

The n -power words $\tilde{w}_{(C,p,n)}$ (Definition 1.2) are utilized to construct a class of nontrivial semigroup identities $\Pi_{(C,p,n)}$. (We use the notation \underline{x} and \underline{y} to mark a specific instance of the variables x and y in a given expression, although these instances stand for the same variables x and y , respectively.)

Construction 1.9. Let $\tilde{w}_{(C,p,n)}$ be a uniform n -power word over $C = \{x, y\}$ such that the words $\tilde{w}_{(C,p,n)} x \tilde{w}_{(C,p,n)}$ and $\tilde{w}_{(C,p,n)} y \tilde{w}_{(C,p,n)}$ are both n -power words over C . Define the 2-variable balanced identity

$$\Pi_{(C,p,n)} : \tilde{w}_{(C,p,n)} \underline{x} \tilde{w}_{(C,p,n)} = \tilde{w}_{(C,p,n)} \underline{y} \tilde{w}_{(C,p,n)}. \tag{1.7}$$

Then, substitute

$$x := \tilde{x} \tilde{y} \quad \text{and} \quad y := \tilde{y} \tilde{x} \tag{1.8}$$

to refine (1.7) to the uniformly balanced identity

$$\check{\Pi}_{(\check{C},p,n)} : \tilde{w}_{(\check{C},p,n)} \tilde{x} \tilde{y} \tilde{w}_{(\check{C},p,n)} = \tilde{w}_{(\check{C},p,n)} \tilde{y} \tilde{x} \tilde{w}_{(\check{C},p,n)}, \tag{1.9}$$

where $\check{C} = \{\tilde{x}, \tilde{y}\}$ and $\tilde{w}_{(\check{C},p,n)}$ is the word obtained from $\tilde{w}_{(C,p,n)}$ by substitution (1.8).

Note that $\tilde{w}_{(\check{C},p,n)}$ needs not be an n -power word over $\check{C} = \{\tilde{x}, \tilde{y}\}$, yet it satisfies law (a) of Definition 1.2. This summarizes Construction 1.9, when a semigroup \mathcal{S} satisfies the identity (1.9), we shortly say that it admits the identity (1.9) by taking $x = uv$ and $y = vu$ for all elements u, v in \mathcal{S} .

Example 1.10 ([14, Example 3.9]). Let $C = \{x, y\}$.

(i) Using the uniform 2-power word $\tilde{w}_{(C,2,2)} = yx^2y^2x$ as in Example 1.3.(i), we receive the identity

$$\Pi_{(C,2,2)} : \quad yx^2y^2x \underline{x} yx^2y^2x = yx^2y^2x \underline{y} yx^2y^2x . \quad (1.10)$$

(ii) Taking the uniform 3-power word $\tilde{w}_{(C,3,3)} = xy^3xyx^3y$ in Example 1.3.(ii), we obtain the identity

$$\Pi_{(C,3,3)} : \quad xy^3xyx^3y \underline{x} xy^3xyx^3y = xy^3xyx^3y \underline{y} xy^3xyx^3y . \quad (1.11)$$

By substituting $x := \tilde{x}\tilde{y}$, $y := \tilde{y}\tilde{x}$, these identities (1.10) and (1.11) become uniformly balanced.

1.5. Semirings and semimodules.

Equipping a set of elements simultaneously by two binary (monoidal) operations, one obtains the following structure.

Definition 1.11. A **semiring** $R := (R, +, \cdot)$ is a set R equipped with two binary operations $+$ and \cdot , called addition and multiplication, such that:

- (i) $(R, +)$ is an Abelian monoid with identity element $0 = 0_R$;
- (ii) (R, \cdot) is a monoid with identity element $1 = 1_R$;
- (iii) Multiplication distributes over addition.

A semiring R is called **idempotent** if $a + a = a$ for every $a \in R$, while it is said to be **bipotent** (or **selective**) if $a + b \in \{a, b\}$ for any $a, b \in R$. When the multiplicative monoid (R, \cdot) is an abelian group, R is called a **semifield**.

Clearly, idempotence implies bipotence, but not conversely. A standard general reference for the structure of semirings is [9].

Remark 1.12. Any ordered monoid (\mathcal{M}, \cdot) gives rise to a semiring, where we define the addition $a + b$ to be $\max\{a, b\}$. Indeed, associativity is clear, and distributivity follows from (1.1). In the same way from an abelian group we obtain a semifield.

Monoid homomorphisms extend naturally to semirings.

Definition 1.13. A **homomorphism** of semirings is a map $\varphi : R \longrightarrow R'$ that preserves addition and multiplication. To wit, φ satisfies the following properties for all a and b in R :

- (i) $\varphi(a + b) = \varphi(a) + \varphi(b)$;
- (ii) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$;
- (iii) $\varphi(0_R) = 0_{R'}$.

A **unital** semiring homomorphism is a semiring homomorphism that preserves the multiplicative identity, i.e., $\varphi(1_R) = 1_{R'}$.

In the sequel, unless otherwise is specified, our homomorphisms are assumed to be unital.

In analogy to the case of rings, one defines module over semiring in a straightforward way.

Definition 1.14. An **R -module** V over a semiring R is an additive monoid $(V, +)$ together with a scalar multiplication $R \times V \rightarrow V$ satisfying the following properties for all $r_i \in R$ and $v, w \in V$:

- (i) $r(v + w) = rv + rw$;
- (ii) $(r_1 + r_2)v = r_1v + r_2v$;
- (iii) $(r_1r_2)v = r_1(r_2v)$;
- (iv) $1_Rv = v$;
- (v) $0_Rv = 0_V = r0_V$.

Modules over semirings are also called **semimodules**.

2. TROPICAL PLACTIC ALGEBRA

One of the central ground objects of the current paper is the following well-known monoid [30]:

Definition 2.1. The **plactic monoid** is the monoid $\mathcal{P}_I := \text{PLC}(\mathcal{A}_I)$, generated by an ordered set of elements $\mathcal{A}_I := \{a_\ell \mid \ell \in I\}$, subject to the equivalence relation \equiv_{gnu} (known as the **elementary Knuth relations** or the **plactic relations**) defined by

$$\begin{aligned} \text{KN1. } a c b &= c a b \quad \text{if } a \leq b < c, \\ \text{KN2. } b a c &= b c a \quad \text{if } a < b \leq c, \end{aligned} \tag{KNT}$$

for all triplets $a, b, c \in \mathcal{A}_n$. Namely, $\mathcal{P}_I := \mathcal{A}_I^* / \equiv_{\text{gnu}}$ with the identity element e . When $|I| = n$ is finite, \mathcal{P}_I is said to be of **rank** n , or **finitely generated**, and is denoted by \mathcal{P}_n .¹

We write PLC_I for $\text{PLC}(\mathcal{A}_I)$ when the alphabet \mathcal{A}_I is arbitrary. Henceforth, we always assume that $e < a_\ell$ for all $a_\ell \in \mathcal{P}_I$. The relation \equiv_{gnu} is a congruence on the free monoid \mathcal{A}_I^* , e.g. see \mathcal{A}_I^* [29], which is also denoted by \equiv_{plc} to indicate its relevance to PLC_I .

We aim for an algebra whose multiplicative structure comprises that of the plactic monoid, and in which the Knuth relations are inferred from the algebraic operations. Towards this goal, we begin with an axillary semigroup structure, to be employed for a construction of such a desirable algebra – a special semiring. (In what follows we let $I \subset \mathbb{N}$ be a nonempty subset.)

Definition 2.2. A **(partial) forward semigroup** $\mathcal{F}_I = \langle f_i \mid i \in I \rangle$ is a pointed semigroup with an absorbing element \mathbf{o} , generated by the (partial) ordered set of elements² $\{f_i \mid i \in I\}$, subject to the axiom

$$\text{FS: } f_j f_i = \mathbf{o} \text{ whenever } f_j > f_i.$$

When I is finite with $|I| = n$, we write \mathcal{F}_n for \mathcal{F}_I .

Note that the semigroup \mathcal{F}_I is not assumed to be a (partial) ordered semigroup, i.e., having an order preserving operation $f_i \geq f_j \Rightarrow f_i f_k \geq f_j f_k$ (Definition 1.4), nor a cancellative semigroup, i.e., $f_k f_i = f_k f_j \Rightarrow f_i = f_j$. Usually, \mathcal{F}_I is non-commutative, since otherwise we would always have $f_i f_j = f_j f_i = \mathbf{o}$, implying that \mathcal{F}_I consists of $\{f_i \mid i \in I\} \cup \{\mathbf{o}\}$.

Remark 2.3. The elements of a forward semigroup \mathcal{F}_I are realized as nondecreasing sequences over $\{f_i \mid i \in I\}$. Then it easy to verify that \mathcal{F}_I naively admits the Knuth relations (KNT), since in \mathcal{F}_I each term of (KNT) equals \mathbf{o} .

Clearly, the forward semigroup \mathcal{F}_I is a very degenerated version of the plactic monoid, but useful for the construction of semirings, as seen below in Construction 2.6.

2.1. Tropical plactic algebra.

We open this section by introducing a new semiring structure – a key object in this paper – that encapsulates the plactic monoid (cf. Theorem 2.8 below).

Definition 2.4. A **tropical plactic algebra** plc_I is a (noncommutative) idempotent semiring $(\text{plc}_I, +, \cdot)$ with multiplicative identity element \mathbf{e} and zero \mathbf{o} , generated by an ordered set of elements² $\{a_i \mid i \in I\}$ subject to the axioms (for every $a \leq b \leq c$ in $\{a_i \mid i \in I\}$):

$$\begin{aligned} \text{PA1: } a &= \mathbf{e} + a; \\ \text{PA2: } ba &= a + b \text{ when } b > a; \\ \text{PA3: } a(b + c) &= ab + c; \\ \text{PA4: } (a + b)c &= a + bc. \end{aligned}$$

When $|I| = n$ is finite we write plc_n for plc_I , and say that plc_n is **finitely generated** and has **rank** n . (For indexing matter, \mathbf{e} is also denoted by a_0 .)

We abbreviate our terminology and also write **tropical plactic algebra** for tropical plactic algebra. Arbitrary elements of plc_I are denoted by Gothic letters $\mathbf{u}, \mathbf{v}, \mathbf{a}$, while a, b, c , are devoted for its generators. When it is clear from the context, we suppress the notation of the indexing set I and write plc for plc_I , for short.

¹In the literature, PLC_I is often assumed to finitely generated, however for large parts of our study are more general and finiteness is not required there.

²For simplicity, we assume a countable set of generators, where the generalization to an arbitrary ordered set of generators is obvious. Moreover, one can also generalize this structure by considering a partially ordered set of generators, but such theory is more involved.

As \mathbf{plc}_I is an idempotent semiring, then $u+u=u$ for all $u \in \mathbf{plc}_I$ (not only for generators $a \in \{a_i \mid i \in I\}$). While Axioms PA1 and PA2 are easily figured out, Axioms PA3 and PA4 are more complicated. To better understand them, they are illustrated by the diagrams:

$$\begin{array}{cc}
 \begin{array}{c} ab \\ ac \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \blacksquare & \square & \square \\ \hline \end{array} & = & \begin{array}{c} ab \\ c \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\
 \begin{array}{c} ac \\ bc \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \blacksquare \\ \hline \square & \square & \square \\ \hline \end{array} & = & \begin{array}{c} a \\ bc \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}
 \end{array}$$

in which the columns stand for products of the letters a, b and c and the rows for the additive terms.

Remark 2.5. Given a product $u = a_{\ell_1} \cdots a_{\ell_m}$ in \mathbf{plc}_I , with $\ell_1, \dots, \ell_m \in I$, we write u^\dagger for the realization of u as a word in the free semigroup \mathcal{A}_I^+ over the set of symbols $\mathcal{A}_I = \{a_i \mid i \in I\}$. Then \mathcal{A}_I^+ – the multiplicative structure of \mathbf{plc}_I – is equipped with a **lexicographic order**, determined by order of the symbol set \mathcal{A}_I . We denote this lexicographic order by $<_{\text{lx}}$.

Troplactic algebras appear naturally as semimodules over a semiring (Definition 1.14), often as a projective semimodule realized as projective matrices, where then the equalities in axioms PA1–PA4 are taken with respect to the appropriate setting. A particular abstract construction of plactic algebra can be obtained by the use of forward semigroups (Definition 2.2).

Construction 2.6. Let $\mathcal{F}_I = \langle f_i \mid i \in I \rangle$ be a forward semigroup and \mathfrak{d} an adjoint distinguished additively idempotent element $\mathfrak{d} = \mathfrak{d} + \mathfrak{d}$ that commutes with each element f_i , i.e., $\mathfrak{d}f_i = f_i\mathfrak{d}$. Set $a_i := \mathfrak{d} + f_i$, whose order is induced by the order \mathcal{F}_I , i.e.,

$$a_i < a_j \iff f_i < f_j,$$

to define a troplactic algebra \mathbf{plc}_I as the algebra generated by a_i . We often take $\mathfrak{d} = \mathfrak{e}$ to be the multiplicative identity of \mathbf{plc}_I .

In this setup, if $a_i < a_j$, then $a_j a_i = \mathfrak{d}(a_i + a_j)$. Indeed, $a_j a_i = (\mathfrak{d} + f_j)(\mathfrak{d} + f_i) = \mathfrak{d}^2 + \mathfrak{d}f_i + \mathfrak{d}f_j = \mathfrak{d}(\mathfrak{d} + f_i + f_j) = \mathfrak{d}(a_i + a_j)$. Thus, for the case $\mathfrak{d} = \mathfrak{e}$, only axioms PA3 and PA4 are enforced, while Axiom PA1 is obtained by construction and Axiom PA2 follows from the structure of \mathcal{F}_I .

Construction 2.6 turns out to be very useful in applications, especially for troplactic matrix algebra, as its arithmetics provides a natural additive decomposition for multiplicative expressions, followed from Axiom PA2. (Note that \mathbf{plc}_I is not a bipotent semiring and its addition is not necessarily determined as maximum.)

Remark 2.7. Let $a, b, c \in \{a_i \mid i \in I\}$ be the generators of \mathbf{plc}_I , and assume that $a \leq b \leq c$.

- (i) Declaring that $\mathfrak{e} < a_i$ for all $i \in I$, Axiom PA1 is then derived directly from Axiom PA2. Furthermore, in this case, the equality $\mathfrak{e} = a + \mathfrak{e}$ implies that $a = \mathfrak{e}$. Indeed $\mathfrak{e} = a + \mathfrak{e} = a\mathfrak{e} = a$.
- (ii) The multiplicative identity \mathfrak{e} is also an additive identity (by Axiom PA1)

$$a\mathfrak{e} = \mathfrak{e}a = a = a + \mathfrak{e} = \mathfrak{e} + a,$$

and in particular it is idempotent with respect to both addition and multiplication, i.e.,

$$\mathfrak{e} + \mathfrak{e} = \mathfrak{e} = \mathfrak{e} \cdot \mathfrak{e}.$$

- (iii) Axiom PA1 implies

$$ab + a = ab + b = ab,$$

since $ab = a(b + \mathfrak{e}) = ab + a$, and $ab = (a + \mathfrak{e})b = ab + b$. Therefore, by idempotence of $u = ab$,

$$ab + a + b = ab,$$

which implies that

$$ab + ba = ab,$$

and furthermore that

$$a^m + a^{m-1} + \cdots + a = a^m. \tag{2.1}$$

Inductively, together with the use of Axiom PA2, for an arbitrary product $a_{\ell_1} \cdots a_{\ell_m}$ we obtain

$$\sum_{S^\dagger \sqsubseteq L_m} \prod_{s \in S^\dagger} a_s = a_{\ell_1} \cdots a_{\ell_m}, \quad L_m = [\ell_1, \dots, \ell_m].$$

Namely, a decomposition of products to sums of non-decreasing terms.

(iv) When $\mathbf{a} = \mathbf{b}$, Axiom PA3 reads as $\mathbf{aa} + \mathbf{ac} = \mathbf{aa} + \mathbf{c}$, and thus by (iii) we have

$$\mathbf{a}^2 + \mathbf{ac} + \mathbf{c}^2 = \mathbf{a}^2 + \mathbf{c} + \mathbf{c}^2 = \mathbf{a}^2 + \mathbf{c}^2, \quad (2.2)$$

hence $(\mathbf{a} + \mathbf{c})^2 = \mathbf{a}^2 + \mathbf{c}^2$.

(v) When $\mathbf{b} = \mathbf{c}$, Axiom PA4 reads as $\mathbf{bb} + \mathbf{ab} = \mathbf{bb} + \mathbf{a}$.

(vi) If $\mathbf{ab} = \mathbf{c}$, then, by Axioms PA1, PA2, and part (iii), we have

$$\mathbf{ba} = \mathbf{b} + \mathbf{a} = \mathbf{b} + \mathbf{c} + \mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{ab} + \mathbf{a} + \mathbf{ab} = \mathbf{ab} + \mathbf{ab} = \mathbf{c} + \mathbf{c} = \mathbf{c},$$

and thus also $\mathbf{ba} = \mathbf{c}$. Accordingly, when $\mathbf{ab} = \mathbf{c}$ for all $\mathbf{b} > \mathbf{a}$, all the elements of \mathbf{plc}_I are of the form \mathbf{a}_ℓ^k , with $k \in \mathbb{N}$.

The main virtue of the troplactic algebra is that the (multiplicative) Knuth relations can be deduced from its semiring structure. Therefore, multiplicatively, \mathbf{plc}_I has the structure of the plactic monoid \mathcal{P}_I (Definition 2.1), which in general is not degenerated.

Theorem 2.8. *The troplactic algebra \mathbf{plc}_I admits the Knuth relations (KNT):*

$$\begin{aligned} \text{KN1 : } & \mathbf{a} \mathbf{c} \mathbf{b} = \mathbf{c} \mathbf{a} \mathbf{b} \quad \text{if } \mathbf{a} \leq \mathbf{b} < \mathbf{c}, \\ \text{KN2 : } & \mathbf{b} \mathbf{a} \mathbf{c} = \mathbf{b} \mathbf{c} \mathbf{a} \quad \text{if } \mathbf{a} < \mathbf{b} \leq \mathbf{c}, \end{aligned}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{\mathbf{a}_i \mid i \in I\}$, and thus multiplicatively forms a plactic monoid.

Proof. We employ Axiom PA1 via Remark 2.7.(ii).

KN1: By axiom PA2:

$$\mathbf{ab} + \mathbf{ac} = \mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{acb},$$

while Axiom PA1 gives (cf. Remark 2.7.(iii))

$$\mathbf{ab} + \mathbf{c} = \mathbf{ab} + \mathbf{b} + \mathbf{c} = \mathbf{ab} + \mathbf{cb} = (\mathbf{a} + \mathbf{c})\mathbf{b} = \mathbf{cab}.$$

Composing both by using Axiom PA3, we obtain

$$\mathbf{cab} = \mathbf{ab} + \mathbf{ac} = \mathbf{ab} + \mathbf{c} = \mathbf{acb},$$

as desired.

KN2: By axiom PA4

$$\mathbf{a} + \mathbf{bc} = \mathbf{bc} + \mathbf{ac} = (\mathbf{b} + \mathbf{a})\mathbf{c} = \mathbf{bac}.$$

On the other hand, Axiom PA1 and Axiom PA2 imply

$$\mathbf{a} + \mathbf{bc} = \mathbf{a} + \mathbf{b} + \mathbf{bc} = \mathbf{ba} + \mathbf{bc} = \mathbf{b}(\mathbf{a} + \mathbf{c}) = \mathbf{bca},$$

yielding the relation $\mathbf{bac} = \mathbf{bca}$.

Thus, \mathbf{plc}_I admits the Knuth relations (KNT), and therefore the congruence \equiv_{knu} . \square

Theorem 2.8 also implies that one can produce a troplactic algebra from a given plactic monoid $\mathcal{P}_I = \mathcal{A}_I / \equiv_{\text{knu}}$ (Definition 2.1) by adjoining a zero element and introducing a formal additive (commutative) operation $+$, subject to Axioms PA2-PA4.

Example 2.9.

- (i) \mathbf{plc}_1 is the **trivial troplactic algebra** has rank 1 and it is generated by a single, and thus $\mathbf{plc}_1 = \{\mathbf{a}^k \mid k \in \mathbb{N}_0\}$.
- (ii) The troplactic algebra \mathbf{plc}_2 with two generators $\mathbf{b} > \mathbf{a}$ consists of the elements of the forms

$$\mathbf{a}^i \mathbf{b}^j, \quad \mathbf{a}^i \mathbf{b}^j + \mathbf{a}^p + \mathbf{b}^q, \quad p > i, q > j,$$

where $i, j, p, q \in \mathbb{N}_0$. If $\mathbf{ab} = \mathbf{c}$, then also $\mathbf{ba} = \mathbf{c}$ by Remark 2.7.(vi), and \mathbf{plc}_2 contains only powers of \mathbf{a} and \mathbf{b} .

In troplactic algebra, by the proof of Theorem 2.8, we see that the terms of the form $\mathbf{acb} = \mathbf{cab} = \mathbf{ab} + \mathbf{c}$ and $\mathbf{bac} = \mathbf{bca} = \mathbf{a} + \mathbf{bc}$ decompose in \mathbf{plc}_I to sums of products of length 1 and 2. On the other hand $\mathbf{cba} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ decomposes to a sum of generators, while \mathbf{abc} is not necessarily decomposable. Thus, in general, the products \mathbf{abc} and \mathbf{cba} in \mathbf{plc}_I are different from $\mathbf{acb} = \mathbf{cab}$ and $\mathbf{bac} = \mathbf{bca}$, and multiplicatively \mathbf{plc}_I has a non-degenerated plactic structure.

In the forthcoming study of troplactic algebra, we intensively use our notations for sequences, cf. §1.1.

Theorem 2.10. Suppose $L_m = [\ell_1, \dots, \ell_m]$ is nondecreasing, with $\ell_1, \dots, \ell_m \in I$ and $m \geq 1$, then

$$\mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_m} = \sum_{S^\dagger \subseteq L_m} \prod_{s \in S^\dagger} \mathbf{a}_s. \quad (2.3)$$

Note that, as L_m is assumed nondecreasing, this means that we only care about maximal nondecreasing sequences and ignore all shorter subsequences. That is, there are no nested nondecreasing sequences.

Proof. Proof by induction on m . The case of $m = 1$ is trivial, and we start $m = 2$. Let $L_2 = [i, j]$, where $i = \ell_1 \leq \ell_2 = j$. Then, by Remark 2.7 we have

$$\sum_{S^\dagger \subseteq L_2} \prod_{s \in S^\dagger} \mathbf{a}_s = \mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_i \mathbf{a}_j = \mathbf{a}_i \mathbf{a}_j.$$

Assuming the implication for $m - 1$, since L_m assumed nondecreasing, we have

$$\begin{aligned} \sum_{S^\dagger \subseteq L_m} \prod_{s \in S^\dagger} \mathbf{a}_s &= \sum_{S^\dagger \subseteq L_{m-1}} \prod_{s \in S^\dagger} \mathbf{a}_s + \left(\sum_{S^\dagger \subseteq L_{m-1}} \prod_{s \in S^\dagger} \mathbf{a}_s \right) \mathbf{a}_{\ell_m} \\ &= \mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_{m-1}} + (\mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_{m-1}}) \mathbf{a}_{\ell_m} \\ &= \mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_{m-1}} (\mathbf{e} + \mathbf{a}_{\ell_m}) = \mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_m}, \end{aligned}$$

as required. \square

Remark 2.11. From Theorem 2.10 we learn that for a nondecreasing sequence (i.e., word) the addition of \mathbf{plc}_I is compatible with the lexicographic order (cf. Remark 2.5),

$$\mathbf{u}^\dagger \leq_{\text{lx}} \mathbf{v}^\dagger \text{ and } \mathbf{u}^\dagger \subseteq_{\text{wd}} \mathbf{v}^\dagger \Rightarrow \mathbf{u} + \mathbf{v} = \mathbf{v}.$$

Note that in general $\mathbf{a} < \mathbf{b}$ does not imply $\mathbf{a} + \mathbf{b} = \mathbf{b}$, and also that the stand alone condition $\mathbf{u}^\dagger \leq_{\text{lx}} \mathbf{v}^\dagger$ does not imply $\mathbf{u} + \mathbf{v} = \mathbf{v}$.

The linkage between inclusion of words to their lexicographic order allow to consider maximality in a suitable algebraic context.

Corollary 2.12. Any product $\mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_m}$ in \mathbf{plc}_I , with $\ell_1, \dots, \ell_m \in I$ and $m \geq 1$, decomposes as

$$\mathbf{a}_{\ell_1} \cdots \mathbf{a}_{\ell_m} = \sum_{S^\dagger \in \text{MSq}^\dagger(L_m)} \prod_{s \in S^\dagger} \mathbf{a}_s, \quad L_m = [\ell_1, \dots, \ell_m]. \quad (2.4)$$

Proof. If $S \subseteq L_m$ is not nondecreasing then $\prod_{s \in S} \mathbf{a}_s$ splits to a sum of terms by Axiom PA2, otherwise for the nondecreasing sequences $S^\dagger \subseteq L_m$ it is enough to restrict to the maximal ones, by Theorem 2.10 and Remark 2.11. \square

In particular cases of strictly decreasing words, this Corollary provides us the following useful computational simplification.

Corollary 2.13. Any finite product $(\mathbf{a}_{\ell_1})^{q_1} \cdots (\mathbf{a}_{\ell_m})^{q_m}$ in \mathbf{plc}_I with $\mathbf{a}_{\ell_1} > \mathbf{a}_{\ell_2} > \cdots > \mathbf{a}_{\ell_m}$ decomposes to

$$(\mathbf{a}_{\ell_1})^{q_1} \cdots (\mathbf{a}_{\ell_m})^{q_m} = \sum_{t=1}^m (\mathbf{a}_{\ell_t})^{q_t}, \quad (2.5)$$

and thus

$$(\mathbf{a}_{\ell_1})^{q_1} \cdots (\mathbf{a}_{\ell_m})^{q_m} + (\mathbf{a}_{\ell_m})^{q_m} \cdots (\mathbf{a}_{\ell_1})^{q_1} = (\mathbf{a}_{\ell_1})^{q_1} \cdots (\mathbf{a}_{\ell_m})^{q_m}.$$

We also have the following generalization to arbitrary finite products.

Corollary 2.14. $\mathbf{u}^m + \mathbf{u}^{m-1} + \cdots + \mathbf{u} = \mathbf{u}^m$ for every finite product $\mathbf{u} \in \mathbf{plc}_I$, i.e., a word \mathbf{u}^\dagger over \mathcal{A}_I .

Proof. For every $k < m$, each term in decomposition (2.4) of \mathbf{u}^k is also contained in one of the terms of decomposition (2.4) of \mathbf{u}^m , both are nondecreasing, and we are done by Theorem 2.10. \square

Recall that the **elementary symmetric polynomials** in n variables X_1, \dots, X_n are defined as

$$\begin{aligned} \text{elm}_k(X_1, \dots, X_n) &= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} X_{i_1} X_{i_2} \cdots X_{i_k} \\ &= \sum_{S_k^\dagger \subseteq N} \prod_{s \in S_k^\dagger} X_s, \end{aligned} \quad N = [1, \dots, n], \quad (2.6)$$

for $k = 1, \dots, n$. In analogy, the **elementary symmetric expressions** on a finite alphabet \mathcal{A}_n are syntactic formulas, defined as

$$\text{elm}_k(\mathcal{A}_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1} a_{i_2} \dots a_{i_k} = \sum_{S_k^\dagger \subseteq N} \prod_{s \in S_k^\dagger} a_s, \quad N = [1, \dots, n], \quad (2.7)$$

for $k = 1, \dots, n$. By Remark 2.7.(iii), for expressions taken over \mathbf{plc}_n , we have

$$\text{elm}_\ell(\mathbf{plc}_n) + \text{elm}_k(\mathbf{plc}_n) = \text{elm}_\ell(\mathbf{plc}_n)$$

for any $\ell \geq k$.

Lemma 2.15. *Any pair of generators \mathbf{a} and \mathbf{b} of \mathbf{plc}_I admits the **Frobenius property**:*

$$(\mathbf{a} + \mathbf{b})^m = \mathbf{a}^m + \mathbf{b}^m \quad \text{for any } m \in \mathbb{N}. \quad (2.8)$$

Proof. Proof by induction on m . The case of $m = 2$ is given by Remark 2.7.(iv):

$$\begin{aligned} (\mathbf{a} + \mathbf{b})^2 &= \mathbf{a}^2 + \mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a} + \mathbf{b}^2 \\ &= \mathbf{a}^2 + \mathbf{a} + \mathbf{b} + \mathbf{a}\mathbf{b} + \mathbf{b}^2 = \mathbf{a}^2 + \mathbf{b}^2. \end{aligned}$$

Assuming the implication holds for $m - 1$, write

$$\begin{aligned} (\mathbf{a} + \mathbf{b})^m &= (\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b})^{m-1} = (\mathbf{a} + \mathbf{b})(\mathbf{a}^{m-1} + \mathbf{b}^{m-1}) \\ &= \mathbf{a}^m + \mathbf{a}\mathbf{b}^{m-1} + \mathbf{b}\mathbf{a}^{m-1} + \mathbf{b}^m \\ &= \mathbf{a}^m + \mathbf{a}\mathbf{b}^{m-1} + \mathbf{b} + \mathbf{a}^1 + \mathbf{a}^2 + \dots + \mathbf{a}^{m-1} + \mathbf{b}^m && [\text{by Theorem 2.10}] \\ &= \mathbf{a}^m + \mathbf{a}\mathbf{b}^{m-1} + \mathbf{b}^m \\ &= \mathbf{a}^m + (\mathbf{a}\mathbf{b} + \mathbf{b}^2)\mathbf{b}^{m-2} && [\text{by Axiom PA3}] \\ &= \mathbf{a}^m + (\mathbf{a} + \mathbf{b}^2)\mathbf{b}^{m-2} = \mathbf{a}^m + \mathbf{a}\mathbf{b}^{m-2} + \mathbf{b}^m \\ &= \mathbf{a}^m + \mathbf{a}\mathbf{b}^{m-2} + \mathbf{b}^{m-1} + \mathbf{b}^m && [\text{by (2.10)}] \\ &= \mathbf{a}^m + (\mathbf{a}\mathbf{b} + \mathbf{b}^2)\mathbf{b}^{m-3} + \mathbf{b}^m \\ &= \mathbf{a}^m + (\mathbf{a} + \mathbf{b}^2)\mathbf{b}^{m-3} + \mathbf{b}^m \\ &\vdots \\ &= \mathbf{a}^m + (\mathbf{a}\mathbf{b} + \mathbf{b}^2) + \mathbf{b}^m \\ &= \mathbf{a}^m + \mathbf{a} + \mathbf{b}^2 + \mathbf{b}^m = \mathbf{a}^m + \mathbf{b}^m \end{aligned}$$

as desired. □

As an immediate consequence of this proof we obtain the following.

Corollary 2.16. *Suppose $\mathbf{a} < \mathbf{b}$ are generators of \mathbf{plc}_I , then*

$$(\mathbf{a} + \mathbf{b})^m = \mathbf{a}^m + \mathbf{b}^m = \mathbf{b}^m \mathbf{a}^m = (\mathbf{b}\mathbf{a})^m \quad (2.9)$$

for any $m \in \mathbb{N}$.

The troplactic algebra \mathbf{plc}_I in general does not admit the Frobenius property (2.8), i.e., for elements that are not generators; for example

$$\mathbf{u} = (\mathbf{a}\mathbf{b} + \mathbf{c})^2 = (\mathbf{a}\mathbf{b})^2 + \mathbf{a}\mathbf{b}\mathbf{c} + \mathbf{c}^2 \neq (\mathbf{a}\mathbf{b})^2 + \mathbf{c}^2 = \mathbf{v},$$

since $\mathbf{a}\mathbf{b}\mathbf{c}$ appear in \mathbf{u} but not in \mathbf{v} .

Remark 2.17. *Not all the elements of \mathbf{plc}_I can be rewritten as products, i.e., \mathbf{plc}_I has elements that are not multiplicatively generated. For example, the element $\mathbf{ad} + \mathbf{bc}$ is not provided as any word \mathbf{u}^\dagger in \mathbf{plc}_I . Indeed, otherwise the terms \mathbf{ad} and \mathbf{bc} would be subwords of \mathbf{u}^\dagger , which does not contain other increasing subwords of length 2, but this impossible.*

2.2. Dual tropical plactic algebra.

We turn to a dual version of a troplactic algebra \mathbf{plc}_I (cf. Definition 2.4), to be employed later, carrying analogous properties.

Definition 2.18. A *dual troplactic algebra*, denoted \mathbf{plc}_I^\wedge , is a (noncommutative) idempotent semiring $(\mathbf{plc}_I^\wedge, \hat{+}, \cdot)$ with (multiplicative) identity element \mathbf{e} and zero \mathbf{o} , generated by an ordered set of elements $\{\mathbf{a}_i \mid i \in I\}$ subject to the axioms (for every $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ in $\{\mathbf{a}_i \mid i \in I\}$):

- PA1': $\mathbf{a} = \mathbf{e} \hat{+} \mathbf{a}$;
- PA2': $\mathbf{ab} = \mathbf{a} \hat{+} \mathbf{b}$ when $\mathbf{b} > \mathbf{a}$;
- PA3': $(\mathbf{b} \hat{+} \mathbf{c})\mathbf{a} = \mathbf{ba} \hat{+} \mathbf{c}$;
- PA4': $\mathbf{c}(\mathbf{b} \hat{+} \mathbf{a}) = \mathbf{a} \hat{+} \mathbf{cb}$.

For indexing matter, we denote \mathbf{e} also by \mathbf{a}_0 .

(We use the special notation “ $\hat{+}$ ” for the addition in \mathbf{plc}_I^\wedge to distinguish it from the addition of \mathbf{plc}_I .)

In analogy to Remark 2.7, by changing nondecreasing sequences by nonincreasing sequences, we now have following properties.

Remark 2.19. Let $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ be elements in the generating set $\{\mathbf{a}_i \mid i \in I\}$ of \mathbf{plc}_I^\wedge .

- (i) Axiom PA1' is derived directly from Axiom PA2' by declaring that $\mathbf{e} < \mathbf{a}_i$ for all i .
- (ii) The multiplicative identity \mathbf{e} is also an additive identity (by Axiom PA1')

$$\mathbf{ae} = \mathbf{ea} = \mathbf{a} = \mathbf{a} \hat{+} \mathbf{e} = \mathbf{e} \hat{+} \mathbf{a}.$$

- (iii) Since $\mathbf{ab} = \mathbf{a}(\mathbf{b} \hat{+} \mathbf{e}) = \mathbf{ab} \hat{+} \mathbf{a}$, and $\mathbf{ab} = (\mathbf{a} \hat{+} \mathbf{e})\mathbf{b} = \mathbf{ab} \hat{+} \mathbf{b}$, by Axiom PA1', the idempotence of \mathbf{plc}_I^\wedge implies that

$$\mathbf{ab} \hat{+} \mathbf{a} \hat{+} \mathbf{b} = \mathbf{ab} \hat{+} \mathbf{a} = \mathbf{ab} \hat{+} \mathbf{b} = \mathbf{ab}.$$

This gives

$$\mathbf{a}^m \hat{+} \mathbf{a}^{m-1} \hat{+} \dots \hat{+} \mathbf{a} = \mathbf{a}^m, \quad (2.10)$$

while using Axiom PA2' inductively for an arbitrary product $\mathbf{a}_{\ell_1} \dots \mathbf{a}_{\ell_m}$ we get

$$\sum_{S^\downarrow \subseteq L_m} \prod_{s \in S^\downarrow} \mathbf{a}_s = \mathbf{a}_{\ell_1} \dots \mathbf{a}_{\ell_m}, \quad L_m = [\ell_1, \dots, \ell_m].$$

- (iv) When $\mathbf{a} = \mathbf{b}$, Axiom PA3' reads $\mathbf{aa} \hat{+} \mathbf{ca} = \mathbf{aa} \hat{+} \mathbf{c}$, and thus by (iii) we have

$$\mathbf{a}^2 \hat{+} \mathbf{ca} \hat{+} \mathbf{c}^2 = \mathbf{a}^2 \hat{+} \mathbf{c} \hat{+} \mathbf{c}^2 = \mathbf{a}^2 \hat{+} \mathbf{c}^2, \quad (2.11)$$

and thus $(\mathbf{a} \hat{+} \mathbf{c})^2 = \mathbf{a}^2 \hat{+} \mathbf{c}^2$.

- (v) When $\mathbf{b} = \mathbf{c}$, Axiom PA4' reads $\mathbf{bb} \hat{+} \mathbf{ba} = \mathbf{a} \hat{+} \mathbf{bb}$.

- (vi) If $\mathbf{ba} = \mathbf{e}$, then, by Axioms PA1', PA2', and part (iii), we have

$$\mathbf{ab} = \mathbf{a} \hat{+} \mathbf{b} = \mathbf{a} \hat{+} \mathbf{e} \hat{+} \mathbf{b} \hat{+} \mathbf{e} = \mathbf{a} \hat{+} \mathbf{ba} \hat{+} \mathbf{b} \hat{+} \mathbf{ba} = \mathbf{ba} \hat{+} \mathbf{ba} = \mathbf{e} \hat{+} \mathbf{e} = \mathbf{e},$$

and thus also $\mathbf{ab} = \mathbf{e}$. This implies that all the elements of \mathbf{plc}_I^\wedge are of the form \mathbf{a}_i^k , with $k \in \mathbb{N}$.

Theorem 2.20. The dual troplactic algebra \mathbf{plc}_I^\wedge admits the Knuth relations (KNT)

$$\begin{aligned} \text{KN1:} \quad & \mathbf{a} \mathbf{c} \mathbf{b} = \mathbf{c} \mathbf{a} \mathbf{b} \quad \text{if} \quad \mathbf{a} \leq \mathbf{b} < \mathbf{c}, \\ \text{KN2:} \quad & \mathbf{b} \mathbf{a} \mathbf{c} = \mathbf{b} \mathbf{c} \mathbf{a} \quad \text{if} \quad \mathbf{a} < \mathbf{b} \leq \mathbf{c}, \end{aligned}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{\mathbf{a}_i \mid i \in I\}$.

Proof. We follow similar arguments as in the proof of Theorem 2.8.

KN1: By axiom PA4'

$$\mathbf{a} \hat{+} \mathbf{cb} = \mathbf{cb} \hat{+} \mathbf{ca} = \mathbf{c}(\mathbf{b} \hat{+} \mathbf{a}) = \mathbf{cab}.$$

On the other hand Axiom PA1' and Axiom PA2' imply

$$\mathbf{a} \hat{+} \mathbf{cb} = \mathbf{a} \hat{+} \mathbf{b} \hat{+} \mathbf{cb} = \mathbf{ab} \hat{+} \mathbf{cb} = (\mathbf{a} \hat{+} \mathbf{c})\mathbf{b} = \mathbf{acb},$$

yielding $\mathbf{cab} = \mathbf{acb}$.

KN2: By axiom PA2':

$$\mathbf{b}\mathbf{a} \hat{+} \mathbf{c}\mathbf{a} = (\mathbf{b} \hat{+} \mathbf{c})\mathbf{a} = \mathbf{b}\mathbf{c}\mathbf{a},$$

while Axiom PA1' gives

$$\mathbf{b}\mathbf{a} \hat{+} \mathbf{c} = \mathbf{b}\mathbf{a} \hat{+} \mathbf{b} \hat{+} \mathbf{c} = \mathbf{b}\mathbf{a} \hat{+} \mathbf{b}\mathbf{c} = \mathbf{b}(\mathbf{a} \hat{+} \mathbf{c}) = \mathbf{b}\mathbf{a}\mathbf{c}.$$

Composing both by using Axiom PA3', we obtain

$$\mathbf{b}\mathbf{c}\mathbf{a} = \mathbf{b}\mathbf{a} \hat{+} \mathbf{c}\mathbf{a} = \mathbf{b}\mathbf{a} \hat{+} \mathbf{c} = \mathbf{b}\mathbf{a}\mathbf{c}$$

as required.

Thus, \mathbf{plc}_n^\wedge admits the Knuth relations (KNT), and therefore the congruence \equiv_{knu} . \square

With similar proofs, adapted to Definition 2.18, the results of §2.1 also hold for dual troplactic algebras, replacing nondecreasing sequences by nonincreasing sequences.

3. TROPICAL MATRICES

The tropical **max-plus semiring** $(\mathbb{T}, \vee, \cdot)$ is the set $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ endowed with the operations of maximum and summation (written in the standard algebraic way),

$$a \vee b := \max\{a, b\}, \quad ab := \underset{\text{sum}}{a + b},$$

serving respectively as the semiring addition and multiplication [40]. (We save the sign “+” for the standard summation, and use “ \vee ” for addition.) This semiring is additively idempotent, i.e., $a \vee a = a$ for every $a \in \mathbb{T}$, in which $0 := -\infty$ is the zero element and $1 := 0$ is the multiplicative unit. (Equivalently one can set $\mathbb{T} := \mathbb{N} \cup \{-\infty\}$, $\mathbb{T} := \mathbb{Z} \cup \{-\infty\}$, or $\mathbb{T} := \mathbb{Q} \cup \{-\infty\}$).

Dually, the **min-plus semiring** $(\mathbb{T}_\wedge, \wedge, \cdot)$ is the set $\mathbb{T}_\wedge := \mathbb{R} \cup \{\infty\}$ equipped with the operations

$$a \wedge b := \min\{a, b\}, \quad ab := \underset{\text{sum}}{a + b},$$

addition and multiplication respectively. The addition \wedge over \mathbb{R} can be written in terms of \vee as $a \wedge b = \frac{ab}{a \vee b}$, or as

$$a \wedge b = (a^{-1} \vee a^{-1})^{-1} = -((-a) \vee (-b)), \quad (3.1)$$

where $-a$ stands the standard negation of $a \in \mathbb{R}$ – the tropical multiplicative inverse of a . (When $a = -\infty$ in (3.1) we implicitly take $-a = \infty$.) We write \mathbb{T} for the max-plus semiring $(\mathbb{T}, \vee, \cdot)$ and \mathbb{T}_\wedge for the min-plus semiring $(\mathbb{T}_\wedge, \wedge, \cdot)$.

The **boolean semiring** is the two element idempotent semiring $\mathbb{B} := (\{0, 1\}, \vee, \cdot)$, whose addition and multiplication are given for $a, b \in \{0, 1\}$ by:

$$a \vee b = \begin{cases} 1 & \text{unless } a = b = 0, \\ 0 & \text{otherwise,} \end{cases} \quad ab = \begin{cases} 0 & \text{unless } a = b = 1, \\ 1 & \text{otherwise.} \end{cases}$$

\mathbb{B} embeds naturally in \mathbb{T} via $0 \mapsto 0$ and $1 \mapsto 1$, while the projection

$$\pi : \mathbb{T} \longrightarrow \mathbb{B}, \quad a \mapsto \begin{cases} 1 & a \neq 0, \\ 0 & a = 0, \end{cases} \quad (3.2)$$

is a semiring homomorphism (Definition 1.13).

3.1. Tropical matrices.

Tropical matrices are matrices with entries in \mathbb{T} , whose multiplication is induced from the operations of \mathbb{T} as in the familiar matrix construction. The set of all $n \times n$ tropical matrices forms a noncommutative semiring, denoted by $\text{Mat}_n(\mathbb{T})$. The **unit** I of $\text{Mat}_n(\mathbb{T})$ is the matrix with $1 := 0$ on the main diagonal and whose off-diagonal entries are all $0 := -\infty$. The zero matrix is denoted by (0) , the absorbing element of $\text{Mat}_n(\mathbb{T})$. Formally, for any nonzero matrix $A \in \text{Mat}_n(\mathbb{T})$ we set $A^0 := I$. A matrix $A \in \text{Mat}_n(\mathbb{T})$ with entries $a_{i,j}$, $i, j = 1, \dots, n$, is written as $A = (a_{i,j})$. For a pair of matrices $A, B \in \text{Mat}_n(\mathbb{T})$, we define

$$A \geq B \quad \text{if } a_{i,j} \geq b_{i,j} \text{ for all } i, j, \quad (3.3)$$

and $A > B$ if $A \geq B$ with $a_{i,j} > b_{i,j}$ for some i, j . Equivalently, we have

$$A \geq B \quad \text{iff} \quad A + B = B.$$

Therefore, in general, $\text{Mat}_n(\mathbb{T})$ is a partially ordered semiring with $(0) < A$ for all A . We denote the monoid of all upper tropical triangular matrices by $\text{TMat}_n(\mathbb{T})$.

Remark 3.1. Given a matrix $A = (a_{i,j})$ in $\text{Mat}_n(\mathbb{T})$ we write $-A$ for the matrix

$$-A := (a_{i,j}^{-1}) = (-a_{i,j}). \quad (3.4)$$

Then, by (3.1), the entry-wise operation gives

$$A \wedge B = -(-A \vee -B),$$

i.e., the entry-wise minimum of the matrices A and B .

The **structure map** is the onto monoid homomorphism

$$\tilde{\pi} : \text{Mat}_n(\mathbb{T}) \longrightarrow \text{Mat}_n(\mathbb{B}) \quad (3.5)$$

sending a tropical matrix $A \in \text{Mat}_n(\mathbb{T})$ to the boolean matrix $\tilde{\pi}(A) \in \text{Mat}_n(\mathbb{B})$ defined as $\tilde{\pi}(A) = (\pi(a_{i,j}))$, by the entry-wise mapping (3.2). (It is a “value forgetful” homomorphism that maps nonzero entries to 1 and zero-entries to 0.) The matrix $\tilde{\pi}(A)$ is called the **structure matrix** of A .

Tropical matrix algebra was systematically studied in its generalized context of supertropical algebra [20], providing many tropical analogues to classical results [13, 21, 22, 23], while subgroup and semigroups structures of these matrices were discussed in [38, 48]. As a matrix monoid, in comparison to matrices over a field, important submonoids of $\text{Mat}_n(\mathbb{T})$ have a very special behavior, as have been dealt in [14, 15, 19], yielding the following results.

Theorem 3.2 ([14, Theorem 4.11]). *The submonoid $\text{TMat}_n(\mathbb{T}) \subset \text{Mat}_n(\mathbb{T})$ of all upper (or dually lower) triangular tropical matrices satisfies the semigroup identities defined in Construction 1.9:*

$$\Pi_{(C,n-1,n-1)} : \tilde{w}_{(C,n-1,n-1)} x \tilde{w}_{(C,n-1,n-1)} = \tilde{w}_{(C,n-1,n-1)} y \tilde{w}_{(C,n-1,n-1)}, \quad (3.6)$$

for $C = \{x, y\}$, with $x = AB$ and $y = BA$.

Theorem 3.3 ([15, Theorem 4.14]). *Any nonsingular subsemigroup $\mathcal{M}_n \subset \text{Mat}_n(\mathbb{T})$ satisfies the identities (3.6) by letting $x = A^{n!} B^{n!}$ and $y = B^{n!} A^{n!}$.*

Clearly, these results hold also to boolean matrices.

We recall some basic definitions from [12, 13, 20, 21].

Definition 3.4. Given a tropical matrix $A = (a_{i,j})$ in $\text{Mat}_n(\mathbb{T})$.

- (i) The **transpose** of A is defined as $A^t = (a_{j,i})$ and satisfies $(AB)^t = B^t A^t$ for any $B \in \text{Mat}_n(\mathbb{T})$.
- (ii) The **permanent** of A is defined as:

$$\text{per}(A) = \bigvee_{\sigma \in \mathfrak{S}_n} \prod_i a_{i,\sigma(i)},$$

where \mathfrak{S}_n denotes the set of all the permutations over $\{1, \dots, n\}$.

- (iii) A is called **nonsingular** if there exist a unique permutation $\tau_A \in \mathfrak{S}_n$ that reaches $\text{per}(A)$; that is, $\text{per}(A) = \prod_i a_{i,\tau_A(i)}$. Otherwise, A is called **singular**.
- (iv) The **(tropical) rank** of A is the largest k for which A has a $k \times k$ nonsingular sub-matrix. (Equivalently, the rank is the maximal number of independent columns (or rows) of A , cf. [20].)
- (v) The **trace** and the **multiplicative trace** of A are defined respectively as

$$\text{tr}(A) = \bigvee_i a_{i,i}, \quad \text{mtr}(A) = \prod_i a_{i,i}.$$

It is easily seen that $\text{per}(A) \geq \text{mtr}(A)$ and $\text{mtr}(AB) \geq \text{mtr}(A) \text{mtr}(B)$, for any matrices $A, B \in \text{Mat}_n(\mathbb{T})$.

3.2. Digraph view to tropical matrices.

Square tropical matrices correspond uniquely to weighted digraphs, whose products interpreted as weights of paths [2], as described below. An $n \times n$ tropical matrix $A = (a_{i,j})$ is associated to the **weighted digraph** $G_A := (\mathcal{V}, \mathcal{E})$ on the vertex set $\mathcal{V} := \{1, \dots, n\}$ with a directed edge $\varepsilon_{i,j} := (i, j) \in \mathcal{E}$ of **weight** $a_{i,j}$ from i to j for every $a_{i,j} \neq 0$. The digraph of the transpose matrix A^t is obtained by redirecting the edges of G_A , i.e., an edge (i, j) is replaced by the edge (j, i) of the same weight.

A **path** γ is a sequence of edges $\varepsilon_{i_1, j_1}, \dots, \varepsilon_{i_m, j_m}$, with $j_k = i_{k+1}$ for every $k = 1, \dots, m-1$. We write $\gamma := \gamma_{i,j}$ to indicate that γ is a path from $i = i_1$ to $j = j_m$. The **length** $\text{len}(\gamma)$ of a path γ is the number of its edges. Formally, we consider also paths of length 0, which we call **empty paths**. The **weight** $\omega(\gamma)$ of a path γ is the tropical product of the weights of all the edges ε_{i_k, j_k} composing γ , counting repeated edges. The weight of an empty path is formally set to be 0.

A path is **simple** if it crosses each vertex at most once – it does not have repeating vertices. (Accordingly, an empty path is considered also as simple.) A (simple) path that starts and ends at the same vertex is called a **(simple) cycle**. An edge $\rho_i := \varepsilon_{i,i}$ is called a **loop**. We write $(\rho)^k$ for the composition $\rho \circ \dots \circ \rho$ of a loop ρ repeated k times, and call it a **multiloop**. The notation $(\rho)^0$ is formal, stands for the empty loop, which can be realized as a vertex. A graph is **acyclic** if it has no cycles of length > 1 (i.e., it may have loops).

In combinatorial view, entries in powers of a tropical matrix $A \in \text{Mat}_n(\mathbb{T})$ correspond to paths of maximal weights in the associated digraph G_A of A . Namely, the (i, j) -entry of the matrix A^m corresponds to the highest weight over all the paths $\gamma_{i,j}$ from i to j of length m in G_A .

The graph view of products $A_{\ell_1} \cdots A_{\ell_m}$ of different $n \times n$ matrices $A_{\ell_t} \in \{A_\ell \mid \ell \in I\}$ is more involved and includes paths with edges contributed by different digraphs $G_{A_\ell} = (\mathcal{V}, \mathcal{E}_\ell)$ occurring on the common vertex set \mathcal{V} . To cope with this generalization, we assign the weighted edges $\varepsilon_{A_\ell} \in \mathcal{E}_\ell$ of each digraph G_{A_ℓ} with a unique color, say c_ℓ , and define the **colored digraph**

$$G_{A_{\ell_1} \cdots A_{\ell_m}} := \bigcup_{t=1}^m G_{A_{\ell_t}} \quad (\ell_t \in I)$$

to have the vertex set $\mathcal{V} = \{1, \dots, n\}$ and a set of edges $\bigcup_{t=1}^m \mathcal{E}_{\ell_t}$ colored c_{ℓ_t} , obtained from $G_{A_{\ell_t}}$. This digraph may have multiple edges with different colors, an edge of $G_{A_{\ell_1} \cdots A_{\ell_m}}$ is denoted by $[\varepsilon_{A_\ell}]_{i,j}$.

In this setting, the (i, j) -entry of a matrix product $A_{\ell_1} \cdots A_{\ell_m}$ corresponds to the highest weight of all colored paths $[\varepsilon_{A_{\ell_1}}]_{i_1, j_1}, \dots, [\varepsilon_{A_{\ell_m}}]_{i_m, j_m}$ of length m from $i = i_1$ to $j = j_m$ in the digraph $G_{A_{\ell_1} \cdots A_{\ell_m}}$, where each edge $[\varepsilon_{A_{\ell_t}}]_{i_t, j_t}$ has color c_{ℓ_t} , $t = 1, \dots, m$. Namely, every edge of the path is contributed uniquely by the digraph $G_{A_{\ell_t}}$, respecting the color ordering determined by the multiplication concatenation $A_{\ell_1} \cdots A_{\ell_m}$. Working with a colored digraph $G_{A_{\ell_1} \cdots A_{\ell_m}}$, we always restrict to colored paths, called **properly colored paths**, that respect the sequence of coloring $c_{\ell_1}, \dots, c_{\ell_m}$. For this reason, we use the awkward notation $G_{A_{\ell_1} \cdots A_{\ell_m}}$ that records the multiplication concatenation $A_{\ell_1} \cdots A_{\ell_m}$.

Notation 3.5. A matrix product $U = A_{\ell_1} \cdots A_{\ell_m}$ is denoted as U^\dagger to indicate that U is realized as the word obtained by the concatenation of the symbols “ A_1 ”, ..., “ A_m ”, accordingly $G_{A_{\ell_1} \cdots A_{\ell_m}}$ is denoted by G_{U^\dagger} , while $U = (u_{i,j})$ stands for the actual result of the matrix product. A path $\gamma_{i,j}$ in G_{U^\dagger} is denoted by $[\gamma_{U^\dagger}]_{i,j}$ to indicate the sequence of its edges’ coloring determined by U^\dagger .

In this notation, an entry $u_{i,j}$ of $U = A_{\ell_1} \cdots A_{\ell_m}$ encodes the highest weight over all properly colored paths $[\gamma_{U^\dagger}]_{i,j}$, determined by the word U^\dagger , of length m from i to j in the colored digraph G_{U^\dagger} . Conversely, the word U^\dagger can be recovered from the coloring sequence of the edges composing any path $\gamma_{i,j}$ of length m in G_{U^\dagger} .

3.3. Synoptic matrices.

We begin with a certain class of tropical matrices of a special characteristic.

Definition 3.6. A **synoptic matrix** is a matrix $A = (a_{i,j})$ in $\text{Mat}_n(\mathbb{T})$ in which

$$a_{i,j} \geq a_{i+1,j} \vee a_{i,j-1}, \quad (3.7)$$

for all $i = 1, \dots, n-1$ and $j = 2, \dots, n$.

We denote the set of all synoptic matrices by $\text{Syn}_n(\mathbb{T})$. By definition, in any synoptic matrix:

- (a) Each row has a nondecreasing order (from left to right);
- (b) Each column has a nonincreasing order (from top to bottom).

In particular, the zero matrix (0) is also synoptic,

Lemma 3.7. $\text{Syn}_n(\mathbb{T})$ is a subsemiring in $\text{Mat}_n(\mathbb{T})$.

Proof. To see that $\text{Syn}_n(\mathbb{T})$ is closed for multiplication, let $C = AB$ and compute $C = (c_{i,j})$ as

$$\begin{aligned} c_{i,j} &= \bigvee_{t=1}^n a_{i,t} b_{t,j} \geq \left(\bigvee_{t=1}^n a_{i,t} (b_{t,j-1} \vee b_{t+1,j}) \right) \vee \left(\bigvee_{t=1}^n (a_{i+1,t} + a_{i,t-1}) b_{t,j} \right) \\ &\geq \left(\bigvee_{t=1}^n a_{i,t} b_{t,j-1} \right) \vee \left(\bigvee_{t=1}^n a_{i+1,t} b_{t,j} \right) = c_{i,j-1} \vee c_{i+1,j}. \end{aligned}$$

Hence property (3.7) is preserved. The verification that $\text{Syn}_n(\mathbb{T})$ is closed for addition is immediate. \square

3.4. Corner and flat matrices.

In the next subsections we develop a theory of special types of tropical matrices, brought here in its full generality, to be used also for future applications.

Notation 3.8. Given a matrix A , we denote by $\text{Row}(A)$ and $\text{Col}(A)$ the set of its rows and columns, respectively. We write $A[I, _]$ for the restriction of A to rows $I \subseteq \text{Row}(A)$ and $A[i, _]$ for the row $\mathbf{r}_i \in \text{Row}(A)$. Similarly, $A[_, J]$ stands for the restriction of A to columns $J \subseteq \text{Col}(A)$, and $A[_, j]$ for the column $\mathbf{c}_j \in \text{Col}(A)$. $A[I, J]$ denotes the restriction of A the rows $I \subseteq \text{Row}(A)$ and to the columns $J \subseteq \text{Col}(A)$. When A is a square matrix we say that $A[I, J]$ is **principal submatrix** if $J = I$.

We start with structure matrices, provided by the structure map $\tilde{\pi} : \text{Mat}_n(\mathbb{T}) \longrightarrow \text{Mat}_n(\mathbb{B})$, cf. (3.5), and consider first the combinatorial shape of matrices. In this setup, we ignore the values of nonzero entries of matrices (these values will show up later) and identify all of them with $1 \in \mathbb{B}$.

Definition 3.9. A **corner matrix** is a matrix whose top-right **corner block** (possibly empty) contains only nonzero entries and all entries out of this block are $\mathbb{0}$. A matrix $A \in \text{Mat}_n(\mathbb{T})$ is called **(p, q)-corner** if its corner-block $B_{\mathbf{p}, \mathbf{q}}$ is $A[I_{\mathbf{p}}, J_{\mathbf{q}}]$, where

$$I_{\mathbf{p}} = \{1, \dots, \mathbf{p}\} \quad \text{and} \quad J_{\mathbf{q}} = \{\mathbf{q}, \dots, n\}, \quad \mathbf{p}, \mathbf{q} \in N := \{1, \dots, n\},$$

i.e., $a_{i,j} \neq \mathbb{0}$ for all $(i, j) \in I_{\mathbf{p}} \times J_{\mathbf{q}}$, otherwise $a_{i,j} = \mathbb{0}$.

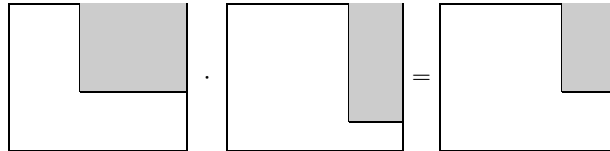
Two corner matrices A and B in $\text{Mat}_n(\mathbb{T})$ are said to be **block-similar**, written $A \sim_{\text{blk}} B$, if $\tilde{\pi}(A) = \tilde{\pi}(B)$, i.e., both are **(p, q)-corner** for some $(\mathbf{p}, \mathbf{q}) \in N \times N$. When $\tilde{\pi}(A) \leq \tilde{\pi}(B)$, we write $A \subseteq_{\text{blk}} B$, which means that the corner block of A is contained in that of B .

The block indexing (\mathbf{p}, \mathbf{q}) indicates the position of the bottom left corner of a block, and thus uniquely determines the corner block. Note that for boolean matrices the block inclusion \subseteq_{blk} is compatible with the matrix (partial) order (3.3), in the sense that $\tilde{\pi}(A) \subseteq_{\text{blk}} \tilde{\pi}(B)$ implies $\tilde{\pi}(A) \leq \tilde{\pi}(B)$.

It easy to verify that the set of all corner matrices forms a multiplicative monoid in $\text{Mat}_n(\mathbb{T})$, which we denote by $\text{Cor}_n(\mathbb{T})$.

Remark 3.10. Suppose $A_{t_u} \in \text{Cor}_n(\mathbb{T})$, $t_u \in N$ with $u = 1, 2, 3$, are $(\mathbf{p}_{t_u}, \mathbf{q}_{t_u})$ -corner matrices, i.e., matrices whose corner blocks are given by $A_{t_u}[I_{t_u}, J_{t_u}]$, with $I_{t_u} = \{1, \dots, \mathbf{p}_{t_u}\}$ and $J_{t_u} = \{\mathbf{q}_{t_u}, \dots, n\}$. Then, letting $s = t_1$, $t = t_2$, and $r = t_3$, we have the following:

- (i) If $\mathbf{p}_s < \mathbf{p}_t$ then $I_s \subset I_t$, while if $\mathbf{q}_s < \mathbf{q}_t$ then $J_s \supset J_t$.
- (ii) If $\mathbf{q}_s > \mathbf{p}_t$, that is $J_s \cap I_t = \emptyset$, then $A_s A_t = (\mathbb{0})$.
- (iii) If $\mathbf{q}_s \leq \mathbf{p}_t$ then $A_s A_t$ is $(\mathbf{p}_s, \mathbf{q}_t)$ -corner, whose corner block is $(A_s A_t)[I_s, J_t]$.



In particular, for any $(\mathbf{p}_t, \mathbf{q}_t)$ -corner matrix A_t with $\mathbf{q}_t \leq \mathbf{p}_t$, $(A_t)^m$ is $(\mathbf{p}_t, \mathbf{q}_t)$ -corner for every $m \in \mathbb{N}$. Otherwise, for $\mathbf{q}_t > \mathbf{p}_t$, we get $(A_t)^m = (\mathbb{0})$.

- (iv) The block inclusion $A_r A_t \subseteq_{\text{blk}} A_r A_s$ holds iff $\mathbf{q}_t \geq \mathbf{q}_s$ (i.e., $J_t \subseteq J_s$), while $A_r A_t \subseteq_{\text{blk}} A_s A_t$ iff $\mathbf{p}_r \leq \mathbf{p}_s$ (i.e., $I_r \subseteq I_t$). In general for arbitrary nonzero products (i.e., $\mathbf{q}_{t_i} \leq \mathbf{p}_{t_{i+1}}$ and $\mathbf{q}_{s_j} \leq \mathbf{p}_{s_{j+1}}$) we have

$$A_{s_1} \cdots A_{s_u} \subseteq_{\text{blk}} A_{t_1} \cdots A_{t_v} \quad \Leftrightarrow \quad \mathbf{p}_{s_1} \leq \mathbf{p}_{t_1} \quad \text{and} \quad \mathbf{q}_{s_u} \geq \mathbf{q}_{t_v}, \quad (3.8)$$

and in particular

$$A_{s_1} \cdots A_{s_u} \sim_{\text{blk}} A_{t_1} \cdots A_{t_v} \quad \Leftrightarrow \quad \mathbf{p}_{s_1} = \mathbf{p}_{t_1} \text{ and } \mathbf{q}_{s_u} = \mathbf{q}_{t_v}. \quad (3.9)$$

In this paper we are especially interested in (\mathbf{p}, \mathbf{q}) -corner matrices obtained as products of (\mathbf{p}, \mathbf{p}) -corner matrices, we call the latter **\mathbf{p} -corner matrices**, for short, and denote them by $A_{(\mathbf{p})}$. These are a special type of triangular (\mathbf{p}, \mathbf{q}) -corner matrices with $\mathbf{p} \geq \mathbf{q}$. However, we have the following general observation, obtained by Remark 3.10.(ii).

Remark 3.11. *The monoid $\text{Cor}_n(\mathbb{T})$ of all (\mathbf{p}, \mathbf{q}) -corner $n \times n$ matrices, ordered by inclusion, forms a partial forward semigroup (Definition 2.2). Restricting to a fixed generating subset of \mathbf{p} -corner matrices with different $\mathbf{p} = 1, \dots, n$, one obtains a finitely generated forward semigroup.*

So far we have concerned only with the structure of corner matrices, rather than the values of their nonzero entries. Considering suitable values for these entries, block inclusions can be translated to matrix inequalities (3.3). To obtain this view we restrict to corner matrices whose nonzero entries all have a same fixed value $\kappa \neq 0$. We call these matrices **κ -flat corner matrices**. For every κ -flat \mathbf{p} -corner matrix $F_{(\mathbf{p})}$ we have

$$(F_{(\mathbf{p})})^m = \kappa^{m-1} F_{(\mathbf{p})}, \quad \mathbf{p} = 1, \dots, n, \quad (3.10)$$

for any $m \in \mathbb{N}$. In particular $F_{(\mathbf{p})}$ is idempotent when $\kappa = 1$.

Lemma 3.12. *Let $F_{(\mathbf{p})}, F_{(\mathbf{q})}, F_{(\mathbf{r})}$ be κ -flat corner matrices with $\mathbf{p} \leq \mathbf{q} \leq \mathbf{r}$ then:*

- (i) $F_{(\mathbf{q})}F_{(\mathbf{p})} = (0)$ for $\mathbf{p} < \mathbf{q}$;
- (ii) $F_{(\mathbf{p})}F_{(\mathbf{q})}$ is (\mathbf{p}, \mathbf{q}) -corner for $\mathbf{p} \leq \mathbf{q}$;
- (iii) $F_{(\mathbf{p})}F_{(\mathbf{q})} > F_{(\mathbf{p})}F_{(\mathbf{r})}$ for $\mathbf{q} < \mathbf{r}$;
- (iv) $F_{(\mathbf{q})}F_{(\mathbf{r})} > F_{(\mathbf{p})}F_{(\mathbf{r})}$ for $\mathbf{p} < \mathbf{q}$.

Proof. Since $\mathbf{p} \leq \mathbf{q} \leq \mathbf{r}$ we have

$$I_{\mathbf{p}} \subseteq I_{\mathbf{q}} \subseteq I_{\mathbf{r}}, \quad J_{\mathbf{r}} \subseteq J_{\mathbf{q}} \subseteq J_{\mathbf{p}}, \quad (*)$$

with strict inclusions when $\mathbf{p} < \mathbf{q} < \mathbf{r}$.

(i) and (ii) are a special case of products of corner matrices, pointed in Remark 3.10 (i) and (iii), respectively.

(iii): By (*) and Remark 3.10.(iv) we have $F_{(\mathbf{p})}F_{(\mathbf{q})} \supseteq_{\text{blk}} F_{(\mathbf{p})}F_{(\mathbf{r})}$, where each nonzero entry of $F_{(\mathbf{p})}F_{(\mathbf{q})}$ and $F_{(\mathbf{p})}F_{(\mathbf{r})}$ equals κ^2 .

(v): Use a similar argument as in (iii), where now $F_{(\mathbf{q})}F_{(\mathbf{r})} \supseteq_{\text{blk}} F_{(\mathbf{p})}F_{(\mathbf{r})}$. \square

3.5. Tropical linear representations.

We write $\mathbb{T}^{(n)}$ for the Cartesian product $\mathbb{T} \times \cdots \times \mathbb{T}$, with \mathbb{T} repeated n times, considered as a \mathbb{T} -module over the semiring \mathbb{T} (Definition 1.14), whose operations induced by the operations of \mathbb{T} . As in this paper we focus on combinatorial aspects, we identify the associative algebra $\text{Lin}(\mathbb{T}^{(n)})$ of all tropical linear operators on $\mathbb{T}^{(n)}$ with $\text{Mat}_n(\mathbb{T})$.

A finite dimensional **tropical (linear) representation** of a semigroup \mathcal{S} , over $\mathbb{T}^{(n)}$, is a semigroup homomorphism

$$\rho : \mathcal{S} \longrightarrow \text{Mat}_n(\mathbb{T}).$$

A representation ρ is said to be **faithful** if it is an injective homomorphism. As in classical representation theory one should think of a representation as a **tropical linear action** of \mathcal{S} on the space $\mathbb{T}^{(n)}$ (since to every $a \in \mathcal{S}$, there is an associated tropical linear operator $\rho(a)$ in $\text{Lin}(\mathbb{T}^{(n)})$ that acts on $\mathbb{T}^{(n)}$). Tropically, these representations have an extra digraph meaning (cf. §3.2), associating a semigroup element to a weighted digraph where the semigroup operation is interpreted as an action of one digraph on another digraph.

Tropical linear representations were introduced in [19], applied there to prove that the bicyclic monoid admits the Adjan identity [1]. These representations are a major method in the present paper, implemented first to a finitely generated forward semigroup (Definition 2.2).

Theorem 3.13. *A finitely generated forward semigroup $\mathcal{F}_n = \langle \mathbf{f}_\ell \mid \ell = 1, \dots, n \rangle$ has a tropical linear representation*

$$\rho : \mathcal{F}_n \longrightarrow \text{TMat}_n(\mathbb{T}), \quad \mathbf{f}_\ell \longmapsto F_{(\ell)}, \quad \mathbf{o} \longmapsto (0),$$

determined by generators' mapping $\mathfrak{f}_\ell \mapsto F_{(\ell)}$, where $F_{(\ell)}$ are ℓ -corner matrices in $\text{TMat}_n(\mathbb{T})$.

Proof. Associativity is clear, while, by Lemma 3.12.(i), $\rho(\mathfrak{f}_j \mathfrak{f}_i) = \rho(\mathfrak{f}_j)\rho(\mathfrak{f}_i) = F_{(j)}F_{(i)} = (0)$ for any $j > i$. \square

Conversely, a finitely generated matrix semigroup $\mathfrak{F}_n = \langle F_{(1)}, F_{(2)}, \dots, F_{(n)} \rangle \subset \text{TMat}_n(\mathbb{T})$, where $F_{(\ell)}$ are κ -flat ℓ -corner matrices of fixed κ , is a forward semigroup, cf. Remark 3.11. Its nonzero elements are all singular matrices of rank 1 (Definition 3.4), and thus \mathfrak{F}_n is a singular matrix subsemigroup of $\text{TMat}_n(\mathbb{T})$, cf. [15].

Corollary 3.14. *The map*

$$\rho_{\text{fw}} : \mathcal{F}_n \longrightarrow \mathfrak{F}_n, \quad \mathfrak{f}_\ell \mapsto F_{(\ell)}, \quad 0 \mapsto (0),$$

determined by generators' mapping $\mathfrak{f}_\ell \mapsto F_{(\ell)}$, is a tropical linear representation of the finitely generated forward semigroup \mathcal{F}_n .

Note that $\rho = \rho_{\text{fw}}$ is not injective, since by (3.10) we have

$$\begin{aligned} \rho(\mathfrak{f}_i \mathfrak{f}_j) &= \rho(\mathfrak{f}_i)\rho(\mathfrak{f}_j) = (F_{(i)})^2 F_{(j)} = \kappa F_{(i)} F_{(j)} \\ &= F_{(i)} \kappa F_{(j)} = F_{(i)} (F_{(j)})^2 = \rho(\mathfrak{f}_i)\rho(\mathfrak{f}_j)\rho(\mathfrak{f}_j) = \rho(\mathfrak{f}_i \mathfrak{f}_j \mathfrak{f}_j), \end{aligned}$$

and thus it is not a faithful representation.

4. TROPLACTIC MATRIX ALGEBRA

In this section we utilize the κ -flat corner matrices to construct a specific troplactic algebra, carrying a digraph meaning, to be utilized later for monoid representations. For matter of generality we use κ as a parameter, assumed taking generic values > 1 , as it also supports a polynomial view to matrix invariants, e.g., traces and permanent.

Our setup consists of the following auxiliary matrices in $\text{TMat}_n(\mathbb{T}) \subset \text{Mat}_n(\mathbb{T})$:

- (a) A collection of κ -flat ℓ -corner matrices $F_{(\ell)} = (f_{i,j}^{(\ell)})$, $\ell = 1, \dots, n$, with fixed κ , defined by

$$f_{i,j}^{(\ell)} = \begin{cases} \kappa & \text{if } i \leq \ell \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

These matrices ordered as $F_{(1)} < F_{(2)} < \dots < F_{(n)}$, and by Lemma 3.12 generate the forward matrix semigroup (Definition 2.2)

$$\mathfrak{F}_n := \langle F_{(1)}, F_{(2)}, \dots, F_{(n)} \rangle, \quad (4.2)$$

cf. (Remark 3.11). (Note that \mathfrak{F}_n is a partially ordered semigroup.)

- (b) A **layout matrix** E that is a triangular idempotent matrix such that $EF_{(\ell)} = EF_{(\ell)}$ for every $\ell = 1, \dots, n$, for simplicity this matrix is taken to be the upper triangular matrix $E = (e_{i,j})$ defined as

$$e_{i,j} = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

Note that E is both multiplicatively and additively idempotent, i.e. $E = E + E = E^2$.

The above matrices are used to define the finitely generated matrix algebra

$$\mathfrak{A}_n := \langle A_{(\ell)} \mid A_{(\ell)} := E \vee F_{(\ell)}, \ell = 1, \dots, n \rangle, \quad (4.4)$$

whose generators are the triangular matrices $A_{(\ell)} := E \vee F_{(\ell)}$, ordered as

$$A_{(1)} < A_{(2)} < \dots < A_{(n)},$$

and has the multiplicative identity E and zero (0) . We write

$$\mathfrak{A}_n^\times := (\mathfrak{A}_n \setminus \{(0)\}, \cdot) \quad (4.5)$$

for the multiplicative monoid of \mathfrak{A}_n . (The whole \mathfrak{A}_n is a pointed multiplicative semigroup, with absorbing element (0) .) The monoid \mathfrak{A}_n^\times is a nonsingular partially ordered monoid, whose members all have rank n , but it is not a cancellative monoid.

We start with the combinatorial structure of the multiplicative monoid \mathfrak{A}_n^\times , that establishes an important linkage to digraphs. The special structure of the generating matrices $A_{(\ell)}$, together with their

digraph realization, allows to record lengths of nondecreasing subsequences of letters (i.e., nondecreasing subwords) by the means of matrix multiplication, leading to the following key lemma.

Key lemma 4.1. *Let $L_m = [\ell_1, \ell_2, \dots, \ell_m]$ be a sequence of indexes $\ell_\xi \in \{1, \dots, n\}$ with $\xi = 1, \dots, m$. Let $U = \prod_{\ell_\xi \in L_m} A_{(\ell_\xi)}$ be a product of generating matrices $A_{(\ell_\xi)}$ in \mathfrak{A}_n^\times , and write $U = (u_{i,j})$. Then, for $i \leq j$, the (i, j) -entry $u_{i,j}$ of the matrix U encodes as the power of κ the length of the longest nondecreasing subsequence of L_m that involves only terms from the (convex) subsequence $[i : j] \subseteq [1 : n]$.*

In other words, the (i, j) -entry $u_{i,j}$ of the matrix U records the length of longest nondecreasing subword of U^\dagger restricted to the convex sub-alphabet “ $A_{(i)}$ ”, ..., “ $A_{(j)}$ ”, cf. Notation 3.5.

Proof. We employ the digraph realization of tropical matrices, cf. §3.2. In this unique realization the digraph $G_{A_{(\ell)}}$ associated to the matrix $A_{(\ell)}$ is an acyclic weighted digraph having the following properties:

- (a) Every vertex $1, \dots, n$ is assigned with a loops of weight $\mathbb{1} = 0$, except the vertex ℓ whose loop ρ_ℓ has weight $\omega(\rho_\ell) = \kappa$.
- (b) The only directed edges of $G_{A_{(\ell)}}$ are
 - (i) $\varepsilon_{s,\ell} = (s, \ell)$ for every $s < \ell$;
 - (ii) $\varepsilon_{\ell,t} = (\ell, t)$ for every $t > \ell$;
 - (iii) $\varepsilon_{s,t} = (s, t)$ only for $s < \ell < t$;
 and all of these matrices have weight κ .

By these properties, given fixed indices $i \leq j$, we see that for all $s < i \leq j < t$ the digraphs $G_{A_{(s)}}$ and $G_{A_{(t)}}$ have no directed edges (p, q) for any $i \leq p < q \leq j$. Therefore, when considering paths between pairs of vertices in $[i : j]$ we can ignore the digraphs $G_{A_{(s)}}$ and $G_{A_{(t)}}$ with $s < i, t > j$.

Let U^\dagger denote the restoring of the multiplication sequence $A_{(\ell_1)} \cdots A_{(\ell_m)}$ as a word (Notation 3.5), and let G_{U^\dagger} denote the graph

$$G_{A_{(\ell_1)} \cdots A_{(\ell_m)}} = \bigcup_{\ell=1}^n G_{A_{(\ell)}},$$

while U stands for the matrix product. Recall from §3.2 that any considered path $\gamma_{i,j}$ in G_{U^\dagger} always respects the edges' coloring determined by the sequence $[\ell_1, \ell_2, \dots, \ell_m]$, $\ell_\xi \in \{1, \dots, n\}$, or equivalently by U^\dagger , and is denoted by $[\gamma_{U^\dagger}]_{i,j}$ to indicate this coloring.

Taking $i \leq j$, an edge $\varepsilon_{i,j}$ from i to j in G_U corresponds to a path $[\gamma_{U^\dagger}]_{i,j}$ of highest weight from i to j of length m in G_{U^\dagger} . By properties (a)-(b) above, we learn that the possible contribution of the digraphs $G_{A_{(s)}}$ and $G_{A_{(t)}}$, $s < i \leq j < t$, to $[\gamma_{U^\dagger}]_{i,j}$ could only be loops, all having weight $\mathbb{1} = 0$. This means, that we can reduce $[\gamma_{U^\dagger}]_{i,j}$ to a sub-path $[\gamma_{U^\dagger}]_{i,j}$ (exactly of the same weight, but perhaps shorter) whose edges coloring is determined by the subsequence $L'_{m'} = [\ell'_1, \dots, \ell'_{m'}] \subseteq [\ell_1, \ell_2, \dots, \ell_m]$, $m' \leq m$, with $\ell'_\xi \in \{i, \dots, j\}$. Writing U'^\dagger for the multiplication sequence $A_{(\ell'_1)} \cdots A_{(\ell'_{m'})}$, realized a word, then $[\gamma_{U'^\dagger}]_{i,j}$ is a colored path from i to j of highest weight and length m' in $G_{U'^\dagger}$.

Let $S^\dagger = [\ell''_1, \ell''_2, \dots, \ell''_{m''}] \subseteq [\ell'_1, \ell'_2, \dots, \ell'_{m'}]$, $\ell''_\xi \in \{i, \dots, j\}$, $m'' \leq m'$, be a longest nondecreasing subsequence of $L'_{m'}$. Let

$$i \leq p_1 < p_2 < \dots < p_r \leq j, \quad r \leq j - i + 1,$$

be the (distinct) elements of $\{i, \dots, j\}$ that take part in S^\dagger , and let q_ξ , $\xi = 1, \dots, r$, be the number of occurrences of p_ξ in S^\dagger . (These occurrences must be sequential occurrences, as S^\dagger is nondecreasing.) Accordingly,

$$\text{len}(S^\dagger) = \sum_{\xi=1}^r q_\xi = m''. \quad (*)$$

We denote by P^\dagger the multiplication sequence $A_{(\ell''_1)} \cdots A_{(\ell''_{m''})}$, determined by S^\dagger and realized as a word. (Namely $P^\dagger \subseteq_{\text{wd}} W^\dagger$.)

Assuming first that $\kappa = 1$, we claim that $\omega(\gamma'_{i,j}) = \text{len}(S^\dagger)$ for $\gamma'_{i,j} := [\gamma_{U'^\dagger}]_{i,j}$. To prove that $\omega(\gamma'_{i,j}) \geq \text{len}(S^\dagger)$, consider the (colored) path

$$[\mu_{P^\dagger}]_{i,j} = (\rho_{p_1})^{q_1-1} \circ \varepsilon_{p_1,p_2} \circ (\rho_{p_2})^{q_2-1} \circ \varepsilon_{p_2,p_3} \circ \dots \circ (\rho_{p_{r-1}})^{q_{r-1}-1} \circ \varepsilon_{p_{r-1},p_r} \circ (\rho_{p_r})^{q_r}$$

of length m'' in G_{P^\dagger} , obtained from P^\dagger by (*). All the edges and loops in $\mu_{i,j} := [\mu_{P^\dagger}]_{i,j}$ have weight 1, and thus $\omega(\mu_{i,j}) = m''$. Clearly, $\omega(\gamma'_{i,j}) \geq \omega(\mu_{i,j})$, since otherwise by plugging in loops of weight 0 we

could expand the path $\mu_{i,j}$ to a properly colored path $\tilde{\mu}_{i,j}$ in G_{U^\dagger} with $\omega(\tilde{\mu}_{i,j}) = \omega(\mu_{i,j})$ to get that $\omega(\tilde{\mu}_{i,j}) > \omega(\gamma'_{i,j})$, contradicting the weight maximality of the path $\gamma'_{i,j}$ in G_{U^\dagger} .

On the other hand, if $\omega(\gamma'_{i,j}) > \omega(\mu_{i,j})$, as $\gamma'_{i,j}$ is an acyclic path, by excluding all loops of weight 0 we could extract from $\gamma'_{i,j}$ a (properly colored) acyclic sub-path $\gamma''_{i,j}$ with $\omega(\gamma''_{i,j}) = \omega(\gamma'_{i,j})$. Then, the coloring of edges in $\gamma''_{i,j}$ determines a nondecreasing sequences in $L'_{m'}$ of length greater than $\omega(\mu_{i,j}) = \text{len}(S^\dagger)$, contradicting the length maximality of S^\dagger . Composing together, we therefore have $\omega(\gamma_{i,j}) = \omega(\gamma'_{i,j}) = \omega(\mu_{i,j}) = \text{len}(S^\dagger)$.

To complete the proof for a generic κ , just observe that $m = 1^m$ and that the map $\kappa^m \mapsto m$, $m \in \mathbb{N}$, is a bijection. \square

Key Lemma 4.1 has dealt with the multiplicative (monoid) structure of \mathfrak{A}_n^\times , and now we turn to discuss the semiring structure of \mathfrak{A}_n .

Theorem 4.2. *The finitely generated matrix algebra $\mathfrak{A}_n := \langle A_{(1)}, \dots, A_{(n)} \rangle$ of (4.4) is a troplactic algebra plc_n (Definition 2.4). Thus, every triplet of it generators*

$$A = A_{(p)}, \quad B = A_{(q)}, \quad C = A_{(r)}, \quad p \leq q \leq r,$$

admit the Knuth relations (KNT):

$$\begin{aligned} \text{KN1 :} \quad & ACB = CAB \quad \text{if } p \leq q < r, \\ \text{KN2 :} \quad & BAC = BCA \quad \text{if } p < q \leq r, \end{aligned}$$

and these matrix products are different from ABC and CBA .

Proof. We verify the axioms of troplactic algebra in Definition 2.4.

PA1: ($\mathbf{a} = \mathbf{e} + \mathbf{a}$.) By construction, and the additive idempotence of $\text{Mat}_n(\mathbb{T})$ we have

$$A_{(p)} = E + F_{(q)} = E + E + F_{(p)} = E + A_{(p)}.$$

PA2: ($\mathbf{ba} = \mathbf{a} + \mathbf{b}$ if $\mathbf{b} > \mathbf{a}$.) By Lemma 3.12.(i) we have

$$\begin{aligned} A_{(q)}A_{(p)} &= (E + F_{(p)})(E + F_{(p)}) = E + F_{(p)} + F_{(q)} \\ &= (E + F_{(p)}) + (E + F_{(q)}) = A_{(p)} + A_{(q)}. \end{aligned}$$

PA3: ($\mathbf{a(b+c)} = \mathbf{ab} + \mathbf{c}$.) By Lemma 3.12.(iii) we have

$$\begin{aligned} A_{(p)}(A_{(q)} + A_{(r)}) &= (E + F_{(p)})(E + F_{(q)} + F_{(r)}) \\ &= E + F_{(p)} + F_{(q)} + F_{(r)} + F_{(p)}F_{(q)} + F_{(p)}F_{(r)} \\ &= E + F_{(p)} + F_{(q)} + F_{(r)} + F_{(p)}F_{(q)} \\ &= (E + F_{(p)} + F_{(q)} + F_{(p)}F_{(q)}) + (E + F_{(r)}) = A_{(p)}A_{(q)} + A_{(r)}. \end{aligned}$$

PA4: ($\mathbf{(a+b)c} = \mathbf{a+bc}$.) By Lemma 3.12.(iv) we have

$$\begin{aligned} (A_{(p)} + A_{(q)})A_{(r)} &= (E + F_{(p)} + F_{(q)})(E + F_{(r)}) \\ &= E + F_{(p)} + F_{(q)} + F_{(r)} + F_{(p)}F_{(r)} + F_{(q)}F_{(r)} \\ &= E + F_{(p)} + F_{(q)} + F_{(r)} + F_{(q)}F_{(r)} \\ &= (E + F_{(p)}) + (E + F_{(q)} + F_{(r)} + F_{(q)}F_{(r)}) = A_{(p)} + A_{(q)}A_{(r)}. \end{aligned}$$

Use Key Lemma 4.1, to see that the products $U = ABC$ and $V = CBA$ are differ from the products $X = ACB$ and $Y = BAC$, which gives $u_{p,r} = \kappa^3$, $v_{p,r} = \kappa$ for the (p, r) -entry of $U = (u_{i,j})$, $V = (v_{i,j})$ respectively, while $x_{p,r} = y_{p,r} = \kappa^2$ for the (p, r) -entry of $X = (x_{i,j})$ and $Y = (y_{i,j})$. \square

The theorem shows that for three letter words $u \equiv_{\text{clk}} v$ iff $u \equiv_{\text{knu}} v$, cf. (5.2), but in general this relation does not hold for words of an arbitrary length, as seen later in Example 6.8.

Proposition 4.3. *Any two generators $A_{(k)}$ and $A_{(\ell)}$ admit the Frobenius property:*

$$(A_{(k)} + A_{(\ell)})^m = A_{(k)}^m + A_{(\ell)}^m.$$

Proof. Immediate by Lemma 2.15, since \mathfrak{A}_n is a troplactic algebra. \square

Let $W = A_{(\ell_1)} \cdots A_{(\ell_m)}$, $W = (w_{i,j})$, be a matrix product in \mathfrak{A}_n^\times , realized as a word W^\dagger over the symbols " $A_{(\ell_1)}$ ", \dots , " $A_{(\ell_m)}$ ". Setting $\kappa = 1$, the multiplicative monoid \mathfrak{A}_n^\times has two main characters:

(i) The additive trace

$$\chi^+ : \mathfrak{A}_n^\times \longrightarrow \mathbb{N}_0, \quad W \longmapsto \text{tr}(W), \quad (4.6)$$

that provides the maximum occurrences of a letter in W^\dagger .

(ii) The multiplicative trace

$$\chi^\times : \mathfrak{A}_n^\times \longrightarrow \mathbb{N}_0, \quad W \longmapsto \text{mtr}(W), \quad (4.7)$$

that gives the length of the matrix word W^\dagger .

Considering \mathbb{N}_0 as the tropical semiring $(\mathbb{N}_0, \vee, \cdot)$, these characters preserve their type, additive and multiplicative respectively,

$$\chi^+(U \vee V) = \chi^+(U) \vee \chi^+(V), \quad \chi^\times(UV) = \chi^\times(U) \chi^\times(V),$$

for any $U, V \in \mathfrak{A}_n^\times$. Note that $\chi^+(W) = \chi^\times(W)$ iff W^\dagger consists of a single letter, where $\chi^+(W) = \chi^\times(W) = 0$ iff $W^\dagger = E$. For χ^+ and χ^\times we also have the properties

$$\chi^+(UV) \leq \chi^+(U) \chi^+(V), \quad \chi^\times(U \vee V) \leq \chi^\times(U) \vee \chi^\times(V),$$

and hence

$$\chi^+(U \vee V) \leq \chi^+(UV), \quad \chi^\times(U \vee V) \leq \chi^\times(UV).$$

The following conclusion is then evident.

Corollary 4.4. *If $A_{(\ell_1)} \cdots A_{(\ell_m)} = A_{(\ell'_1)} \cdots A_{(\ell'_m)} \neq E$, $\ell_t, \ell'_s \in \{1, \dots, n\}$, then $m = m'$.*

5. THE CLOAKTIC AND THE CO-CLOAKTIC MONOIDS

We gently coarser the Knuth relations (KNT) to introduce a new monoid structure over the alphabet $\mathcal{A}_I = \{a_\ell \mid \ell \in I\}$ whose underlying equivalence is based on the forthcoming function. Given a word $w \in \mathcal{A}_I$ and a convex set $J \subseteq_{\text{cx}} I$ (Definition 1.1), we define the set $\text{sw}_J(w)$ of all subwords u of w with letters in the convex sub-alphabet $\mathcal{A}_J \subseteq_{\text{cx}} \mathcal{A}_I$, i.e.,

$$\text{sw}_J(w) := \{u \subseteq_{\text{wd}} w \mid u \in \mathcal{A}_J^+\}, \quad J \subseteq I \text{ is convex},$$

and specify the subset $\overrightarrow{\text{sw}}_J(w)$ consisting of all nondecreasing words in $\text{sw}_J(w)$. The function $\overrightarrow{\text{len}}_J(w)$ gives that length of the longest nondecreasing word $u \in \overrightarrow{\text{sw}}_J(w)$, possibly non-unique, that is

$$\overrightarrow{\text{len}}_J(w) := \max\{\text{len}(u) \mid u \in \overrightarrow{\text{sw}}_J(w)\}. \quad (5.1)$$

Relying upon on this function, rather than on recombining the underlying relations of \equiv_{kmu} (e.g., as in the Chinese monoid [5]), we define the following new monoid, slightly coarsening the plactic monoid (Definition 2.1).

Definition 5.1. *The **cloaktic monoid** is the monoid $\mathcal{K}_I := \text{CLK}(\mathcal{A}_I)$, generated by an ordered set of elements ³ $\mathcal{A}_I := \{a_\ell \mid \ell \in I\}$, subject to the equivalence relation \equiv_{clk} , defined as*

$$u \equiv_{\text{clk}} v \iff \overrightarrow{\text{len}}_J(u) = \overrightarrow{\text{len}}_J(v) \text{ for all convex subsets } J \subseteq_{\text{cx}} I. \quad (5.2)$$

Namely $\mathcal{K}_I := \mathcal{A}_I^* / \equiv_{\text{clk}}$. When $|I| = n$ is finite, we write \mathcal{K}_n for \mathcal{K}_I and say that \mathcal{K}_n is finitely generated of **rank** n .

We write CLK_I for $\text{CLK}(\mathcal{A}_I)$ when the alphabet \mathcal{A}_I is arbitrary. Henceforth, we always assume that $e < a_\ell$ for all $a_\ell \in \mathcal{K}_I$.

Lemma 5.2. *The equivalence relation \equiv_{clk} is a congruence on the free monoid \mathcal{A}_I^* .*

Proof. Suppose that $u_1 \equiv_{\text{clk}} v_1$ and $u_2 \equiv_{\text{clk}} v_2$, and assume that $u_1 u_2 \not\equiv_{\text{clk}} v_1 v_2$. Then, say, $\overrightarrow{\text{len}}_J(u_1 u_2) > \overrightarrow{\text{len}}_J(v_1 v_2)$ for some $\mathcal{A}_J \subseteq_{\text{cx}} \mathcal{A}_I$. Let with $w \subseteq_{\text{wd}} u_1 u_2$ a longest nondecreasing subword of $u_1 u_2$ with letters from \mathcal{A}_J . As such w decomposes as $w = w_1 w_2$, where $w_1 \subseteq_{\text{wd}} u_1$, $w_2 \subseteq_{\text{wd}} u_2$ are nondecreasing. Accordingly $w_1 \in \mathcal{A}_{J_1}^*$, $w_2 \in \mathcal{A}_{J_2}^*$ such that $\mathcal{A}_{J_1} \cap \mathcal{A}_{J_2} \subseteq \mathcal{A}_J$ where

$$\mathcal{A}_{J_1} \cap \mathcal{A}_{J_2} = \{a_\ell\} \text{ for some letter } a_\ell \in \mathcal{A}_J, \text{ or } \mathcal{A}_{J_1} \cap \mathcal{A}_{J_2} = \emptyset. \quad (*)$$

³For simplicity, we assume a countable set of generators, where the generalization to an arbitrary ordered set of generators is obvious. Moreover, one can also generalize this monoid by considering a partially ordered set of generators, but such theory is more involved.

Since $u_1 \equiv_{\text{clk}} v_1$ and $u_2 \equiv_{\text{clk}} v_2$, there are nondecreasing subwords $w'_1 \subseteq_{\text{wd}} v_1$ in $\mathcal{A}_{J_1}^*$ and $w'_2 \subseteq_{\text{wd}} v_2$ in $\mathcal{A}_{J_2}^*$ satisfying $\text{len}(w'_1) = \text{len}(w_1)$ and $\text{len}(w'_2) = \text{len}(w_2)$. Then $(*)$ ensures that the concatenation $w'_1 w'_2$ is a nondecreasing subword of $u_2 v_2$ and therefore has length $\text{len}(w_1) \text{len}(w_2)$, and hence $\overrightarrow{\text{len}}_J(u_1 u_2) = \overrightarrow{\text{len}}_J(v_1 v_2)$ – a contradiction. \square

When $u \equiv_{\text{clk}} v$, for $u, v \in \mathcal{A}_I^*$, we say that the words u and v are **cloaktically equivalent**. That is $u \equiv_{\text{clk}} v$, if over any convex sub-alphabet \mathcal{A}_J of \mathcal{A}_I the length of longest nondecreasing subwords in u and v is the same. In particular each letter $a_\ell \in \mathcal{A}_I$ is cloaktically equivalent only to itself.

Example 5.3. The **bicyclic monoid** is the monoid $\mathcal{B} := \mathcal{A}_2^* / \equiv_{\text{bcy}}$ generated by two ordered elements $a < b$, subject to the relation \equiv_{bcy} determined by $ab = e$. As well known (e.g. [6]), each element of $w \in \mathcal{B}$ can be canonically written as

$$w = b^i a^j, \quad \text{for unique } i, j \geq 0.$$

Accordingly, for any $w \in \mathcal{B}$ we have

$$\overrightarrow{\text{len}}_{\{1,2\}}(w) = \max \{ \overrightarrow{\text{len}}_{\{1\}}(w), \overrightarrow{\text{len}}_{\{2\}}(w) \}.$$

Thus, we see that on \mathcal{A}_2^* the congruence \equiv_{bcy} implies \equiv_{knu} .

By Definition 5.1 we have the following obvious properties.

Properties 5.4. For two cloaktically equivalent words $u \equiv_{\text{clk}} v$ in \mathcal{K}_I the below properties hold.

- (i) Each letter $a_i \in \mathcal{A}_I$ appears in u and in v exactly the same times, i.e., the formal relation $u \equiv_{\text{clk}} v$ is balanced (cf. §1.4).
- (ii) The total length of u and v is the same, i.e., $\text{len}(u) = \text{len}(v)$.
- (iii) If u is nondecreasing, or nonincreasing, then $u = v$.
- (iv) The equivalence $u \equiv_{\text{clk}} v$ does not imply that set theoretically $\overrightarrow{\text{sw}}_J(u) = \overrightarrow{\text{sw}}_J(v)$, $J \subseteq_{\text{cx}} I$, nor even that longest words in $\overrightarrow{\text{sw}}_J(u)$ and $\overrightarrow{\text{sw}}_J(v)$ are the same.

Recall that \equiv_{plc} denotes congruence determined by the plactic relations, which are the Knuth relations, and hence \equiv_{plc} is exactly \equiv_{knu} .

Remark 5.5. The monoid homomorphism $\mathcal{A}_I^* \longrightarrow \mathcal{K}_I$ factors through the plactic monoid (Definition 2.1)

$$\mathcal{A}_I^* \twoheadrightarrow \mathcal{A}_I^* / \equiv_{\text{plc}} \twoheadrightarrow \mathcal{A}_I^* / \equiv_{\text{clk}},$$

as will be seen later in Proposition 6.9 and Corollary 6.10. Furthermore, from this factorization, we can conclude that elements $w \in \mathcal{K}_2$ of the cloaktic monoid of rank 2 have a canonical form

$$w = b^i a^j b^k, \quad i \leq j, \quad i, j, k \in \mathbb{N}_0,$$

generalizing Example 5.3.

5.1. Linear representations of the cloaktic monoid.

The tropplactic matrix algebra $\mathfrak{A}_n = \langle A_{(1)}, \dots, A_{(n)} \rangle$, as defined in (4.4), is now utilized to introduce a linear representation of the finitely generated cloaktic monoid $\mathcal{K}_n = \langle a_1, \dots, a_n \rangle$ of rank n (Definition 5.1).

Theorem 5.6. The map

$$\mathfrak{U} : \mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times, \quad a_\ell \longmapsto A_{(\ell)}, \quad \ell = 1, \dots, n, \quad (5.3)$$

defined by mapping of generators, i.e.,

$$\mathfrak{U}(w) = \mathfrak{U}(a_{\ell_1}) \cdots \mathfrak{U}(a_{\ell_m}), \quad w = a_{\ell_1} \cdots a_{\ell_m} \in \mathcal{A}_n^*,$$

and sends the empty word e to E , is a monoid isomorphism – a faithful linear representation of the cloaktic monoid \mathcal{K}_n of rank n .

Proof. Let $L_m = [\ell_1, \ell_2, \dots, \ell_m]$ and $L'_m = [\ell'_1, \ell'_2, \dots, \ell'_m]$ be two sequences with $\ell_t, \ell'_t \in \{1, \dots, n\}$. Write $u = a_{\ell_1} \cdots a_{\ell_m}$ and $v = a_{\ell'_1} \cdots a_{\ell'_m}$ for the corresponding words in \mathcal{K}_n and consider their linear representations

$$\begin{aligned} U &= \mathfrak{U}(u) = \mathfrak{U}(a_{\ell_1}) \cdots \mathfrak{U}(a_{\ell_m}) = A_{(\ell_1)} \cdots A_{(\ell_m)}, \\ V &= \mathfrak{U}(v) = \mathfrak{U}(a_{\ell'_1}) \cdots \mathfrak{U}(a_{\ell'_m}) = A_{(\ell'_1)} \cdots A_{(\ell'_m)}, \end{aligned}$$

written as $U = (u_{i,j})$ and $V = (v_{i,j})$.

By Key Lemma 4.1, the (i,j) -entry $u_{i,j}$, $i \leq j$, of the matrix U gives the length of the longest nondecreasing subsequence of L_m that involves terms only from the convex subsequence $[i : j] \subseteq [1 : n]$. The same holds for the (i,j) -entry $v_{i,j}$ with respect to $L'_{m'}$. Taking the generators corresponding to L_m and $L'_{m'}$ in \mathcal{K}_n , by the definition of the congruence \equiv_{clk} , it implies that $u \equiv_{\text{clk}} v$ iff $U = V$. \square

Corollary 5.7. *The cloaktic monoid \mathcal{K}_n satisfies the Knuth relations (KNT).*

Proof. $\mathcal{U} : \mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times$ is isomorphism by Theorem 5.6, where \mathfrak{A}_n^\times satisfies the Knuth relations (KNT) by Theorem 4.2. \square

Corollary 5.8. *The cloaktic monoid \mathcal{K}_n admits all the semigroup identities satisfied by $\text{TMat}_n(\mathbb{T})$, in particular the semigroup identities (3.6).*

Proof. \mathcal{K}_n is faithfully represented by \mathfrak{A}_n^\times – a submonoid of $\text{TMat}_n(\mathbb{T})$ – which by Theorem 3.2 satisfies the identities (3.6). \square

The map \mathcal{U} does not record explicitly longest convex subwords, but only their lengths. However, by Key Lemma 4.1, we see that each diagonal entry $u_{\ell,\ell}$ of $U = \mathcal{U}(w)$ records precisely the number of times that the letter a_ℓ appears in w , and thus also the total length of w as the multiplicative trace (4.7). The additive trace (4.6) gives the maximal occurrence of a letter,

Lemma 5.9. *Let $T_n \subset \mathcal{A}_n^*$ be the subset of all nondecreasing words over the alphabet \mathcal{A}_n . Then the restriction*

$$\mathcal{U}|_{T_n} : T_n \longrightarrow \mathfrak{A}_n^\times,$$

of the homomorphism (5.3) to T_n is a bijective map.

Proof. A (nondecreasing) word $w \in T_n$ is written uniquely as

$$w = a_1^{q_1} \cdots a_n^{q_n}, \quad q_k \in \mathbb{N}_0.$$

Let $U = \mathcal{U}(w)$, $U = (u_{i,j})$, be the image of w in \mathfrak{A}_n^\times . By Key Lemma 4.1 we have

$$u_{1,k} = \sum_{t=1}^k q_t, \quad \text{for every } k = 1, \dots, n,$$

and therefore the q_k 's are uniquely computed in terms of the entries of U as

$$q_1 = u_{1,1} \quad \text{and} \quad q_k = u_{1,k} - u_{1,k-1} \quad \text{for } k = 2, \dots, n.$$

(Alternatively, $q_k = u_{k,k}$, for every $k = 1, \dots, n$.) \square

Example 5.10. *We provide the full details of the tropical linear representation*

$$\mathcal{U} : \mathcal{K}_3 \longrightarrow \mathfrak{A}_3^\times, \quad a_\ell \longmapsto A_{(\ell)}, \quad \ell = 1, 2, 3,$$

of the cloaktic monoid $\mathcal{K}_3 = \text{CLK}(\mathcal{A}_3)$ of rank 3, given by the generators' map

$$a_1 \longmapsto A = \begin{bmatrix} 1 & 1 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \quad a_2 \longmapsto B = \begin{bmatrix} 0 & 1 & 1 \\ & 1 & 1 \\ & & 0 \end{bmatrix}, \quad a_3 \longmapsto C = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 1 \end{bmatrix},$$

where $A = A_{(1)}$, $B = A_{(2)}$, $C = A_{(3)}$, and $\kappa = 1$. (Recall that $1 := 0$, and the empty space stand for $0 := -\infty$.) For these matrix generators we obtain

$$ACB = CAB = \begin{bmatrix} 1 & 2 & 2 \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \quad BAC = BCA = \begin{bmatrix} 1 & 1 & 2 \\ & 1 & 2 \\ & & 1 \end{bmatrix},$$

where for pairs of generators we have

$$ABA = BAA = \begin{bmatrix} 2 & 2 & 2 \\ & 1 & 1 \\ & & 0 \end{bmatrix}, \quad BBA = BAB = \begin{bmatrix} 1 & 2 & 2 \\ & 2 & 2 \\ & & 0 \end{bmatrix},$$

$$ACA = CAA = \begin{bmatrix} 2 & 2 & 2 \\ & 0 & 1 \\ & & 1 \end{bmatrix}, \quad CCA = CAC = \begin{bmatrix} 1 & 1 & 2 \\ & 0 & 2 \\ & & 2 \end{bmatrix},$$

$$BCB = CBB = \begin{bmatrix} 0 & 2 & 2 \\ & 2 & 2 \\ & & 1 \end{bmatrix}, \quad CCB = CBC = \begin{bmatrix} 0 & 1 & 2 \\ & 1 & 2 \\ & & 2 \end{bmatrix},$$

while

$$ABC = \begin{bmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{bmatrix}, \quad CBA = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}.$$

Finally, by Corollary 5.8, the cloaktic monoid \mathcal{K}_3 of rank 3 satisfies the semigroup identities (3.6), in particular (1.10):

$$\Pi_{(C,2,2)} : \quad yx^2y^2x \underline{x} \ yx^2y^2x = yx^2y^2x \underline{x} \ yx^2y^2x$$

by letting $x = uv$ and $y = vu$, for any $u, v \in \mathcal{K}_3$.

Remark 5.11. Let $\vec{w} = a_{\ell_1} \cdots a_{\ell_m}$ be a word in the free monoid \mathcal{A}_n^* , represented in \mathfrak{A}_n^\times by the matrix product $A_{(\ell_1)} \cdots A_{(\ell_m)}$, via the homomorphism $\mathcal{U} : \mathcal{A}_n^* \longrightarrow \mathfrak{A}_n^\times$, cf. (5.3). The reversal \overleftarrow{w} of the word \vec{w} is represented in \mathfrak{A}_n^\times by the product

$$\mathcal{U} : \overleftarrow{w} \longmapsto ((A_{(\ell_1)})^t \cdots (A_{(\ell_m)})^t)^t.$$

Indeed, by standard composition of transpositions we have:

$$\begin{aligned} \mathcal{U}(\overleftarrow{w}) &= A_{(\ell_m)} \cdots A_{(\ell_1)} = ((A_{(\ell_m)})^t)^t \cdots ((A_{(\ell_1)})^t)^t \\ &= ((A_{(\ell_1)})^t \cdots (A_{(\ell_m)})^t)^t. \end{aligned}$$

In digraph view, cf. §3.2, the transposition of a matrix $A = (a_{i,j})$ is interpreted as the redirecting of all edges in the associated digraph $G_A := (\mathcal{V}, \mathcal{E})$ of A . Namely, replacing each edge $\varepsilon_{i,j} \in \mathcal{E}$ by the edge $\varepsilon_{j,i}$ with opposite direction, but with the same weight. Thus, by Key Lemma 4.1, it means that $\mathcal{U}(\overleftarrow{w})$ gives the lengths of longest nonincreasing subwords in \vec{w} over the convex sub-alphabets of \mathcal{A}_n .

Remark 5.12. Two words $u, v \in \mathcal{K}_n$ can be cloaktically equivalent, i.e., $u \equiv_{\text{clk}} v$, while their reversals \overleftarrow{u} and \overleftarrow{v} are not cloaktically equivalent, or vice versa. For example, take the words $u = c \ b^2 c^2 \ a^2 b^2 c$ and $v = c^2 \ b^2 c \ a^2 b^2 c$ for which $\overleftarrow{u} = cb^2a^2c^2b^2c$ and $\overleftarrow{v} = cb^2a^2cb^2c^2$. Hence $u \equiv_{\text{clk}} v$ but $\overleftarrow{u} \not\equiv_{\text{clk}} \overleftarrow{v}$.

The example in the remark is a pathological example, and it will have further uses in the paper.

Observation 5.13. Let $u, v \in \mathcal{K}_n$ be elements represented in \mathfrak{A}_n^\times by $U = \mathcal{U}(u)$, $V = \mathcal{U}(v)$, and consider the matrix sum $W = U + V$. Since by Theorem 5.6 the representation $\mathcal{U} : \mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times$, cf. (5.3), is isomorphism then $w = \mathcal{U}^{-1}(W)$ is word in \mathcal{K}_n that contains u and v as subwords, in which the lengths of longest nondecreasing subwords (5.1) satisfy

$$\overrightarrow{\text{len}}_J(w) \leq \overrightarrow{\text{len}}_J(u) + \overrightarrow{\text{len}}_J(v), \quad \overrightarrow{\text{len}}_J(w) = \max\{\overrightarrow{\text{len}}_J(u), \overrightarrow{\text{len}}_J(v)\},$$

for every convex sub-alphabet $\mathcal{A}_J \subseteq_{\text{cx}} \mathcal{A}_n$.

From this observation it follows that the additive monoid of \mathfrak{A}_n has the structure of a lattice.

5.2. Algorithm and complexity.

We utilize our faithful representation of the cloaktic monoid (Theorem 5.6, §5.1) to provide the following efficient algorithm. By “roughly similar length” we mean lengths that are equal up to ± 1 .

Algorithm 5.14. Given a word $w \in \mathcal{A}_n^+$ over a finite alphabet $\mathcal{A}_n = \{a_1, \dots, a_n\}$, find the maximal lengths of its nondecreasing subwords over each convex sub-alphabet $\mathcal{A}_J \subseteq_{\text{cx}} \mathcal{A}_n$.

- (1) Decompose w into two subwords of roughly similar length, repeat this decomposition as long as possible.
- (2) Represent each letter a_ℓ of w by the matrix $A_{(\ell)}$ as in (4.4).
- (3) Roll back the recursive decomposition of step (1) where in each backward step multiply the resulting matrices obtained from the previous step.

For complexity considerations, to be compatible with the customarily notation, we switch notation and denote the number of letters in the underlining alphabet \mathcal{A}_m by m ; the length of the input (i.e., the length of the input word) is then denoted by n . We assume that \mathcal{A}_m is a finite alphabet.

Time complexity. *Algorithm 5.14 has a linear time complexity $O(n)$.*

Proof. Reading an input word $w \in \mathcal{A}_m^+$ of length n over a finite alphabet \mathcal{A}_m , is performed in linear time. Computing the product of two $m \times m$ κ -flat corner matrices requires a constant time of order m^2 . As the algorithm is recursive, we have $\log(n)$ matrix multiplications, which in total takes $m^2 \log(n)$ operations. Putting all together, as m is constant we have $O(\log(n))$ effective operations and reading in $O(n)$, which sum up to $O(n)$. \square

To the best of our knowledge, no other known algorithm offers such efficient time complexity for this problem.

5.3. The co-cloaktic monoid.

In the view of Remark 1.7, we employ the co-mirror of words (Definition 1.6) to introduce the following monoid – a second coarsening of the plactic monoid.

Definition 5.15. *The **co-cloaktic monoid** of rank n is the monoid ${}^{\text{co}}\mathcal{K}_n := {}^{\text{co}}\text{CLK}(\mathcal{A}_n)$ generated by a finite ordered set of elements $\mathcal{A}_n := \{a_1, \dots, a_n\}$, subject to the equivalence relation $\equiv_{\text{clk}}^{\text{co}}$, defined by (1.3) as*

$$u \equiv_{\text{clk}}^{\text{co}} v \iff {}^{\text{co}}\mathfrak{M}_n(v) \equiv_{\text{clk}} {}^{\text{co}}\mathfrak{M}_n(u).$$

Namely ${}^{\text{co}}\mathcal{K}_n := \mathcal{A}_n^* / \equiv_{\text{clk}}^{\text{co}}$.

We write ${}^{\text{co}}\text{CLK}_n$ for ${}^{\text{co}}\text{CLK}(\mathcal{A}_n)$ when the alphabet \mathcal{A}_n is arbitrary. As the relation $\equiv_{\text{clk}}^{\text{co}}$ is defined in terms of \equiv_{clk} , which is a congruence (Lemma 5.2), respected by the co-mirroring operation (1.4), then it is a congruence. In general the equivalence \equiv_{clk} on \mathcal{A}_n^* does not implies the equivalence $\equiv_{\text{clk}}^{\text{co}}$, i.e., $u \equiv_{\text{clk}} v$ does not imply $u \equiv_{\text{clk}}^{\text{co}} v$ on \mathcal{A}_n^* , or vice versa. This is shown below in Example 7.15, after developing additional methods.

Properties 5.16. *For any two words $u \equiv_{\text{clk}}^{\text{co}} v$ in ${}^{\text{co}}\mathcal{K}_n$ we have the following properties:*

- (i) *Every letter $a_i \in \mathcal{A}_n$ appears in u and in v exactly the same times, i.e., the formal relation $u \equiv_{\text{clk}}^{\text{co}} v$ is balanced (cf. §1.4).*
- (ii) *The length of u and v is the same.*

As the co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$ arises from the equivalence of the cloaktic monoid \mathcal{K}_n , we manipulate the linear representation of \mathcal{K}_n , cf. §5.1, to construct a representation of the (finitely generated) co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$. That is, ${}^{\text{co}}\mathcal{K}_n = \langle a_1, \dots, a_n \rangle$ of rank n is represented by the monoid homomorphism (determined by generators' mapping)

$$\Omega_\kappa : {}^{\text{co}}\mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times, \quad a_\ell \longmapsto {}^{\text{co}}\mathfrak{M}_n(A_\ell), \quad e \longmapsto E, \quad (5.4)$$

where $\mathfrak{A}_n := \langle A_{(1)}, \dots, A_{(n)} \rangle$ is the troplactic matrix algebra defined in (4.4). Since \mathfrak{A}_n is a troplactic algebra (Theorem 4.2), then Corollary 2.13 implies the explicitly additive decomposition

$$M_{(\ell)} := {}^{\text{co}}\mathfrak{M}_n(A_\ell) = \prod_{\substack{t=n, \\ t \neq n-\ell+1}}^1 A_{(t)} = \bigvee_{\substack{t=1, \\ t \neq n-\ell+1}}^n A_{(t)}. \quad (5.5)$$

Then $M_{(\ell)}$ can be written as

$$M_{(\ell)} = \kappa \check{A}_{(\ell)},$$

where $\check{A}_{(\ell)}$ is the $n \times n$ matrix of the form

$$\check{A}_{(\ell)} = \begin{pmatrix} 1 & \cdots & 1 & 1 & & \cdots & 1 \\ & \ddots & \vdots & \vdots & & & \vdots \\ & & 1 & 1 & & & \\ & & & \kappa^{-1} & 1 & \cdots & 1 \\ & & & & 1 & \cdots & 1 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 1 \end{pmatrix}, \quad \ell = 1, \dots, n, \quad (5.6)$$

whose (ℓ', ℓ') -entry, $\ell' = n - \ell + 1$, is κ^{-1} and its other nonzero entries are all $\mathbb{1}$.

Letting $(\mathfrak{A}_n^\times)^{[1]} := \langle M_{(\ell)}, \dots, M_{(\ell)} \rangle$ be the matrix submonoid of \mathfrak{A}_n^\times generated by the matrices $M_{(\ell)}, \dots, M_{(\ell)}$ in (5.5) and with identity E , from (5.4) we draw the surjective homomorphism

$$\Omega_\kappa^{[1]} : {}^{\text{co}}\mathcal{K}_n \longrightarrow (\mathfrak{A}_n^\times)^{[1]}, \quad a_\ell \mapsto M_{(\ell)}, \quad e \mapsto E, \quad (5.7)$$

by restricting the image of Ω_κ to $(\mathfrak{A}_n^\times)^{[1]}$.

Remark 5.17. Using the endomorphism $\partial_1 : \mathcal{A}_n^* \longrightarrow (\mathcal{A}_n^*)^{[1]}$ in Remark 1.7, the co-cloactic monoid ${}^{\text{co}}\mathcal{K}_n := \mathcal{A}_n^* / \equiv_{\text{clk}}^{\text{co}}$ can be defined equivalently as $u \equiv_{\text{clk}}^{\text{co}} v \Leftrightarrow \partial_1(v) \equiv_{\text{clk}} \partial_1(u)$, and thus $(\mathcal{A}_n^*)^{[1]} / \equiv_{\text{clk}}$ can be realized as a monoid isomorphic to a submonoid of ${}^{\text{co}}\mathcal{K}_n$. (Note that $(\mathcal{A}_n^*)^{[1]} / \equiv_{\text{clk}}$ and $\mathcal{K}_n^{[1]}$ are not necessarily equal as monoids.) On the other hand, $\mathfrak{U} : \mathcal{K}_n \xrightarrow{\sim} \mathfrak{A}_n^\times$, $a_\ell \mapsto A_{(\ell)}$, is an isomorphism by Theorem 5.6, and its composition with the surjection ∂_1 shows that $\Omega_\kappa : {}^{\text{co}}\mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times$, cf. (5.4), is an injective homomorphism. Hence, the map $\Omega_\kappa^{[1]} : {}^{\text{co}}\mathcal{K}_n \longrightarrow (\mathfrak{A}_n^\times)^{[1]}$ is an isomorphism. The following diagram summarizes these homomorphisms

$$\begin{array}{ccccc} \mathcal{A}_n^* & \twoheadrightarrow & \mathcal{K}_n := \mathcal{A}_n^* / \equiv_{\text{clk}} & \xrightarrow{\mathfrak{U}} & \mathfrak{A}_n^\times \\ & \searrow & \downarrow \Omega_\kappa & \nearrow \Omega_\kappa^{[1]} & \downarrow \partial_1 \\ & & {}^{\text{co}}\mathcal{K}_n := \mathcal{A}_n^* / \equiv_{\text{clk}}^{\text{co}} & \xrightarrow{\Omega_\kappa^{[1]}} & (\mathfrak{A}_n^\times)^{[1]} \\ & & \downarrow \Omega & \nearrow \vartheta & \downarrow \vartheta \\ & & & & {}^\diamond\mathfrak{A}_n^\times \end{array} \quad \circlearrowright \quad (5.8)$$

Yet, we need to complete the lower part of the diagram – the object ${}^\diamond\mathfrak{A}_n^\times$.

Note also that by Remark 5.5 and Diagram (5.8), the monoid homomorphism $\mathcal{A}_n^* \longrightarrow {}^{\text{co}}\mathcal{K}_n$ factors through the plactic monoid

$$\mathcal{A}_n^* \twoheadrightarrow \mathcal{A}_n^* / \equiv_{\text{plc}} \twoheadrightarrow \mathcal{A}_n^* / \equiv_{\text{clk}}^{\text{co}}.$$

We will return to this factorization later.

Despite (5.4) provides a faithful representation of ${}^{\text{co}}\mathcal{K}_n$ in terms of monoids, i.e., as a matrix submonoid of \mathfrak{A}_n^\times , the images $\Omega(a_\ell)$ of the generators a_ℓ of ${}^{\text{co}}\mathcal{K}_n$ do not generate the image $\Omega({}^{\text{co}}\mathcal{K}_n)$ as a tropolactic matrix algebra. To achieve this attribute we define the matrix algebra

$${}^\diamond\mathfrak{A}_n := \langle \check{A}_{(1)}, \dots, \check{A}_{(n)} \rangle \subset \text{Mat}_n(\mathbb{T}), \quad (5.9)$$

generated by the matrices $\check{A}_{(1)}, \dots, \check{A}_{(n)}$ introduced in (5.6), which we call the **co-algebra** of \mathfrak{A}_n . The multiplicative submonoid ${}^\diamond\mathfrak{A}_n^\times$ of ${}^\diamond\mathfrak{A}_n$ is called the **co-monoid** of \mathfrak{A}_n^\times , for which we obtain the monoid homomorphism

$$\Omega : {}^{\text{co}}\mathcal{K}_n \longrightarrow {}^\diamond\mathfrak{A}_n^\times, \quad a_\ell \mapsto \check{A}_{(\ell)}, \quad e \mapsto E, \quad (5.10)$$

i.e., $\Omega(a_\ell) = \kappa^{-1} \Omega_\kappa(a_\ell)$ for every $a_\ell \in \mathcal{A}_n$. Accordingly we see that

$$\vartheta : (\mathfrak{A}_n^\times)^{[1]} \xrightarrow{\sim} {}^\diamond\mathfrak{A}_n^\times, \quad M_{(\ell)} \mapsto \check{A}_{(\ell)} (= \kappa^{-1} M_{(\ell)}), \quad (5.11)$$

is a monoid isomorphism, and the composition

$$\mathfrak{D} := \vartheta \circ \partial_1 : \mathfrak{A}_n^\times \longrightarrow {}^\diamond\mathfrak{A}_n^\times, \quad A_{(\ell)} \mapsto \check{A}_{(\ell)}, \quad (5.12)$$

is a surjective homomorphism.

Applying the negation map (3.4) in Remark 3.1,

$$\mathfrak{N} : \check{A}_{(\ell)} \mapsto \hat{A}_{(\ell)} := -\check{A}_{(\ell)}, \quad \check{a}_{i,i} \mapsto -(\check{a}_{i,i}) = (\check{a}_{i,i})^{-1},$$

due to the special structure of $\check{A}_{(\ell)}$, it only replaces the entry $\check{a}_{\ell',\ell'} = \kappa^{-1}$ by $\hat{a}_{\ell',\ell'} := \kappa$, where $\ell' = n - \ell + 1$. The images matrices $\hat{A}_{(\ell)}$ are then considered as matrices in $\text{Mat}_n(\mathbb{T}_\wedge)$, whose multiplication are now induced by the semiring operations $+$ and \wedge . Nevertheless, as in the sequel we want be consistent with the multiplicative operation of the monoid \mathfrak{A}_n^\times , we stick with the matrices $\check{A}_{(\ell)}$ for which the matrix multiplications are taken with respect to semiring operations $+$ and \vee (jointly with negation), while the additive operation $A \wedge B$ can be computed dually as $-(-A \vee -B)$.

Example 5.18. The co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_3$ of rank 3 is linearly represented by the homomorphism $\Omega : {}^{\text{co}}\mathcal{K}_3 \longrightarrow {}^{\diamond}\mathfrak{A}_3^{\times}$ determined by the generators' mapping

$$a_1 \longmapsto \check{A}_{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & & \kappa^{-1} \end{bmatrix}, \quad a_2 \longmapsto \check{A}_{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ \kappa^{-1} & 0 & 0 \\ & & 0 \end{bmatrix}, \quad a_3 \longmapsto \check{A}_{(3)} = \begin{bmatrix} \kappa^{-1} & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}.$$

(Recall that $\mathbb{1} := 0$, and the empty space stand for $\emptyset := -\infty$.) For these generators we have the products

$$\check{A}_{(1)}\check{A}_{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ \kappa^{-1} & 0 & 0 \\ & & \kappa^{-1} \end{bmatrix}, \quad \check{A}_{(2)}\check{A}_{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ \kappa^{-1} & \kappa^{-1} & 0 \\ & & \kappa^{-1} \end{bmatrix}, \quad \check{A}_{(3)}\check{A}_{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ \kappa^{-1} & 0 & 0 \\ & & \kappa^{-1} \end{bmatrix}.$$

Note that $\check{A}_{(1)}\check{A}_{(2)} = \check{A}_{(1)} \wedge \check{A}_{(2)}$, $\check{A}_{(3)}\check{A}_{(1)} = \check{A}_{(3)} \wedge \check{A}_{(1)}$, but $\check{A}_{(2)}\check{A}_{(1)} \neq \check{A}_{(2)} \wedge \check{A}_{(1)}$

Considering the digraph view of the matrix $W = \Omega(w)$ assigned to a word $w \in \mathcal{A}_n^*$, cf. §3.2, a nonzero entry $w_{i,j}$ of W corresponds to a path $[\gamma_{W^\dagger}]_{i,j}$ of maximal (nonpositive) weight from i to j of length $\text{len}(w)$ in G_{W^\dagger} (Notation 3.5). In terms of words, it means the length of the complement of subword of w over $\{a_i, \dots, a_j\}$ of maximal length where each letter has (negative) length κ^{-1} .

Theorem 5.19. The matrix co-algebra ${}^{\diamond}\mathfrak{A}_n := \langle \check{A}_{(1)}, \dots, \check{A}_{(n)} \rangle$ is a dual tropolactic algebra \mathfrak{plc}_n^\wedge (Definition 2.18), and thus every triplet of matrices

$$\check{A} = \check{A}_{(p)}, \quad \check{B} = \check{A}_{(q)}, \quad \check{C} = \check{A}_{(r)}, \quad p < q < r,$$

admit the Knuth relations (KNT).

Proof. We verify the axioms of Definition 2.18 by a straightforward technical computation. By construction

$$p' = n - p + 1, \quad q' = n - q + 1, \quad r' = n - r + 1,$$

are the only κ^{-1} entries in \check{A} , \check{B} , and \check{C} , with $p' > q' > r'$. All other nonzero entries are $\mathbb{1}$.

- (a) The product $\check{A}\check{B}$ has exactly two κ^{-1} entries, the (q', q') -entry and the (p', p') -entry, where all other nonzero entries are $\mathbb{1}$.
- (b) The product $\check{B}\check{A}$ has at least two κ^{-1} entries, the (q', q') -entry and the (p', p') -entry, all other nonzero entries are $\mathbb{1}$. In general, these are the only κ^{-1} entries, except in the case that $p' = q' + 1$, in which also the (q', p') -entry has value κ^{-1} . (See e.g. Example 5.18.)

The other two letter products involving \check{C} have the analogues properties.

PA1': $(\mathbf{a} = \mathbf{c} \hat{+} \mathbf{a})$ Immediate, by comparison of entries.

PA2': $(\mathbf{ab} = \mathbf{a} \hat{+} \mathbf{b} \text{ when } \mathbf{b} > \mathbf{a})$ A direct implication of property (a), since $p' > q'$, which is also the structure of the matrix $\check{A} \wedge \check{B}$.

PA3': $((\mathbf{b} \hat{+} \mathbf{c})\mathbf{a} = \mathbf{ba} \hat{+} \mathbf{c})$ By property (b) the κ^{-1} entries of $\check{B}\check{A} \wedge \check{C}\check{A}$ are (r', r') , (q', q') , and (p', p') , and possibly (r', p') or (q', p') . If the (q', p') -entry in $\check{B}\check{A}$ is κ^{-1} then it is also κ^{-1} in $\check{B}\check{A} \wedge \check{C}\check{A}$, independently on $\check{C}\check{A}$. If the (r', p') -entry is κ^{-1} , then the (q', p') -entry is also κ^{-1} and hence $\check{B} = \check{C}$. Thus, in all cases, $(\check{B} \wedge \check{C})\check{A} = \check{B}\check{A} \wedge \check{C}\check{A}$.

PA4': $(\mathbf{c}(\mathbf{b} \hat{+} \mathbf{a}) = \mathbf{a} \hat{+} \mathbf{cb})$ By property (b) the κ^{-1} entries of $\check{C}\check{B} \wedge \check{C}\check{A}$ are (r', r') , (q', q') , and (p', p') , and possibly (r', p') or (r', q') . If the (r', q') -entry in $\check{C}\check{B}$ is κ^{-1} then it is also κ^{-1} in $\check{C}\check{B} \wedge \check{C}\check{A}$, independently on $\check{C}\check{A}$. If the (r', p') -entry is κ^{-1} , then the (r', q') -entry is also κ^{-1} , which implies $\check{A} = \check{B}$. Thus, in all cases, $\check{C}(\check{B} \wedge \check{A}) = \check{A} \wedge \check{C}\check{B}$.

The proof is then completed by Theorem 2.20. \square

Theorem 5.20. The map

$$\Omega : {}^{\text{co}}\mathcal{K}_n \longrightarrow {}^{\diamond}\mathfrak{A}_n^{\times}, \quad \Omega : a_\ell \longmapsto \check{A}_{(\ell)}, \quad e \longmapsto E, \quad (5.13)$$

defined by generators' mapping, i.e.,

$$\Omega(w) = \Omega(a_{\ell_1}) \cdots \Omega(a_{\ell_m}), \quad w = a_{\ell_1} \cdots a_{\ell_m} \in \mathcal{A}_n^*,$$

is a monoid isomorphism – a faithful linear representation of the co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$ of rank n .

Proof. The monoid homomorphism $\Omega_\kappa : {}^{\text{co}}\mathcal{K}_n \longrightarrow \mathfrak{A}_n^{\times}$ is injective by Remark 5.17, providing the isomorphism $\Omega_\kappa^{[1]} : {}^{\text{co}}\mathcal{K}_n \xrightarrow{\sim} (\mathfrak{A}_n^{\times})^{[1]}$, which translates to $\Omega : {}^{\text{co}}\mathcal{K}_n \longrightarrow {}^{\diamond}\mathfrak{A}_n^{\times}$ by the isomorphism (5.11). \square

By this theorem we see that $u \equiv_{\text{clk}}^{\text{co}} v$ in ${}^{\text{co}}\mathcal{K}_n$ implies that $\Omega(u) \equiv_{\text{knu}} \Omega(v)$ in ${}^{\diamond}\mathfrak{A}_n^{\times}$, but the converse does not hold as seen later in Example 7.14.

Corollary 5.21. *The co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$ satisfies the Knuth relations (KNT).*

Proof. $\Omega : {}^{\text{co}}\mathcal{K}_n \longrightarrow {}^{\diamond}\mathfrak{A}_n^{\times}$ is an isomorphism, by Theorem 5.20, where ${}^{\diamond}\mathfrak{A}_n^{\times}$ by itself admits the Knuth relations by Theorem 5.19. \square

Corollary 5.22. *The co-cloaktic monoid ${}^{\text{co}}\mathcal{K}_n$ admits all the semigroup identities satisfied by $\text{TMat}_n(\mathbb{T})$, in particular the semigroup identities (3.6).*

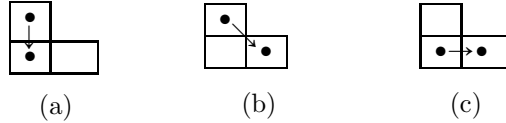
Proof. ${}^{\text{co}}\mathcal{K}_n$ is faithfully represented by ${}^{\diamond}\mathfrak{A}_n^{\times}$ – a submonoid of $\text{TMat}_n(\mathbb{T})$ – which by Theorem 3.2 satisfies the identity (3.6). \square

6. CONFIGURATION TABLEAUX

Towards the goal of introducing linear representations of the plactic monoid \mathcal{P}_I (Definition 2.1), we need to establish its precise linkage to the cloaktic monoid \mathcal{K}_n . To do so we utilize the known bijective correspondence of \mathcal{P}_I to Young tableaux [43], therefore by this study we also obtain a useful algebraic description of the latter combinatorial objects. To simplify indexing notations we denote arbitrary letters in \mathcal{A}_I by x, y , and z .

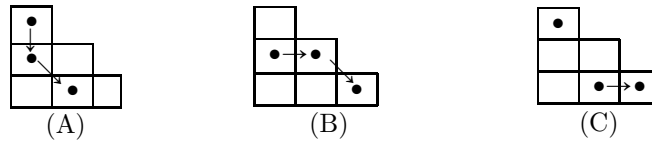
6.1. Young tableaux.

A **Young diagram** (also called Ferrers diagram) is a finite collection of cells, arranged in left-justified rows, such that the number of cells at each row (i.e., the row's length) is less or equal than its predecessor. In this paper we use the France convention in which the longest row is the bottom row, and rows are enumerated from bottom to top [30]. Over this setting, we define three types of (proper) steps from a cell to its neighbor:



- (a) a **vertical step** – a move from a cell to its neighbor below;
- (b) a **slant step** – a move from a cell to its bottom right neighbor;
- (c) a **horizontal step** – a move from a cell to its right neighbor.

These steps are combined to walks and covers in a Young diagram:



- (A) a **falling walk** is a continues composition of sequential vertical steps or slant steps;
- (B) a **descending walk** is a continues composition of sequential horizontal steps or slant steps;
- (C) a **horizontal cover** is a collection of cells, one from each column, taken from a nonincreasing subset of rows.

(Note that all of these are “non-left oriented”.) A **proper walk** is either a falling walk or a descending walk. This terminology is consistent with the content of the Young diagram, as described next. *In what follows, all walks and covers are assumed to be proper.*

A **Young tableau** is a Young diagram whose cells are filled by symbols – a single letter in each cell – taken from a given alphabet \mathcal{A}_I , usually required to be finite and totally ordered.

Definition 6.1. *The **track** $\text{trk}(\gamma)$ of a (proper) walk γ in a Young tableau \mathfrak{T} is the word defined by concatenating the symbols appearing in cells along γ . The **length** $\text{len}(\gamma)$ of a walk γ is the number of cells along γ .*

*When the content of cells in \mathfrak{T} are numbers, the **weight** of γ , denoted $w(\gamma)$, is defined as the sum of cells' values along γ .*

While a walk between two cells needs not be unique, except when the walk is along the bottom row or the left column, all the tracks between pair of cells always have the same length; different tracks may have a same weight.

Young tableaux are combinatorial structure, used intensively in group theory and representation theory [8]. They provide a useful machinery that led to an elegant algorithmic solution (Schensted 1961) for the following problem:

Problem. *Given a (finite) word $w \in \mathcal{A}_I^+$ on the (totally) ordered alphabet \mathcal{A}_I , find the length of its longest nondecreasing subwords of w .*

The central advantage of Schensted's method is that it avoids a precise identification of a longest nondecreasing subword, and presents a useful linkage between semigroups and Young tableaux, as summarized below.

A nondecreasing word $w \in \mathcal{A}_I^+$ is called a **row**, while a (strictly) decreasing word is called a **column**. We say the row $u = x_1 \cdots x_s$ **dominates** the row $v = y_1 \cdots y_t$, written $u \triangleright v$, if $s \leq t$ and $x_i > y_i$ for all $i = 1, \dots, s$. Any word $w \in \mathcal{A}_I^+$ has a unique factorization $w = \mathbf{r}_p \cdots \mathbf{r}_1$ as a concatenation of rows \mathbf{r}_i of maximal length.

Definition 6.2. A **semi-standard tableau** \mathfrak{T} is a tableau that corresponds (by rows) to a word $w \in \mathcal{A}_I^+$ such that $\mathbf{r}_p \triangleright \mathbf{r}_{p-1} \triangleright \cdots \triangleright \mathbf{r}_1$, where \mathbf{r}_1 is the bottom row and \mathbf{r}_p is the top row of \mathfrak{T} . The set of all semi-standard tableaux is denoted by $\text{Tab}(\mathcal{A}_I)$, written Tab_I for short.

We formally adjoin the **empty tableau**, denoted \mathfrak{T}_0 , considered as a semi-standard tableau.

Namely, a tableau is semi-standard if the entries in each row are nondecreasing and the entries in each column are increasing. In the sequel we assume that I is finite and write Tab_n for Tab_I . A tableau is called **standard** if the entries in each row and in each column are increasing, which implies that each letter may appear only once. The subset of all standard tableaux over a finite alphabet \mathcal{A}_n is denoted by STab_n .

We denote the columns of a given tableau \mathfrak{T} by $\mathbf{c}_1, \dots, \mathbf{c}_q$, enumerated from left to right. Reading the columns from bottom to top, each column \mathbf{c}_{j+1} is increasing but not necessarily dominates the column \mathbf{c}_j unless, \mathfrak{T} is standard. In latter case, we define the **transpose tableau** \mathfrak{T}^t of \mathfrak{T} by writing the columns as rows, i.e., $\mathbf{c}_q \cdots \mathbf{c}_1$, for which $(\mathfrak{T}^t)^t = \mathfrak{T}$.

The nonincreasing sequence $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of the rows' length λ_i of a tableau \mathfrak{T} defines the **shape** of \mathfrak{T} and introduces a **partition** of the integer $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, which in its turn determines the graphical representation of the corresponding Young diagram. (The shape of the empty tableau is set to be 0.)

Schensted's algorithm associates each word $w \in \mathcal{A}_n^+$ with a tableau $\mathfrak{T} = \text{tab}(w)$, via the map ⁴

$$\text{tab} : \mathcal{A}_n^+ \longrightarrow \text{Tab}_n,$$

defined inductively as $\text{tab}(wx) = \text{tab}(\text{tab}(w)x)$ for an arbitrary $x \in \mathcal{A}_n$. It is recursively defined in terms of tableaux as

$$\text{tab}(\mathfrak{T}x) = \begin{cases} \mathfrak{T}x & \text{if } \mathbf{r}_1 x \text{ is a row,} \\ \text{tab}(\mathbf{r}_p \cdots \mathbf{r}_2 y) \mathbf{r}'_1 & \text{if } \text{tab}(\mathbf{r}_1 x) = y \mathbf{r}'_1. \end{cases} \quad (6.1)$$

where $\mathbf{r}_p \cdots \mathbf{r}_1$ are the row decomposition of \mathfrak{T} , y is the left most letter of \mathbf{r}'_1 which is strictly greater than x , and \mathbf{r}'_1 is obtained from \mathbf{r}_1 by replacing y with x .

In a more lingual terms, the map tab can be described in algorithmic terms.

Algorithm 6.3 (Bumping Algorithm). *Given a word $w = x_1 \cdots x_m$, $x_i \in \mathcal{A}_n$, start with $\{x_1\}$, which is a Young tableau. Suppose x_1, \dots, x_k have already been inserted, and \mathfrak{T} is the current tableau. To insert x_{k+1} , start with the first row of \mathfrak{T} and search for the first letter which is greater than x_{k+1} . If there is no such element, append x_{k+1} to the end of first row. If there is such an element (say, x_j), exchange x_j by x_{k+1} , and proceed inductively to insert x_j to the next row.*

For three letters $x < y < z$ in \mathcal{A}_n^+ , the algorithm produces the familiar tableaux:

$$\text{tab}(xzy) = \text{tab}(zxy) = \begin{array}{|c|c|} \hline z & \\ \hline x & y \\ \hline \end{array}, \quad \text{tab}(yxz) = \text{tab}(yzx) = \begin{array}{|c|c|} \hline y & \\ \hline x & z \\ \hline \end{array}.$$

⁴In the literature, the map tab is also denoted by P .

For two distinct letters $x < y$ it gives:

$$\text{tab}(xyx) = \text{tab}(yxx) = \begin{array}{|c|c|} \hline y & \\ \hline x & x \\ \hline \end{array}, \quad \text{tab}(xyy) = \text{tab}(yyx) = \begin{array}{|c|c|} \hline y & \\ \hline x & y \\ \hline \end{array}.$$

One immediately sees that the upper tableaux are just realization of the Knuth relations (KNT).

Theorem 6.4 (Schensted 1961). *The length of a longest nondecreasing (reps. decreasing) subword of w is equal to the length of the bottom row \mathbf{r}_1 (resp. height of the first column \mathbf{c}_1) of $\text{tab}(w)$.*

This tableau form introduces on \mathcal{A}_n^+ the equivalence relation

$$u \equiv_{\text{tab}} v \Leftrightarrow \text{tab}(u) = \text{tab}(v).$$

We call $\text{tab}(w)$ the **(canonical) tableau form** of the word $w \in \mathcal{A}_n^+$, and write

$$\mathfrak{T}_w := \text{tab}(w).$$

Conversely, every tableau $\mathfrak{T} \in \text{Tab}_n$ can be identified with $\mathfrak{T}_w = \text{tab}(w)$ for some (not necessarily unique) word $w \in \mathcal{A}_n^+$. Therefore, we may think of \mathfrak{T}_w also as a word having a canonical form in \mathcal{A}_n^+ . The extension of $\text{tab} : \mathcal{A}_n^+ \rightarrow \text{Tab}_n$ to

$$\text{tab} : \mathcal{A}_n^* \longrightarrow \text{Tab}_n$$

is defined naturally by sending $e \mapsto \mathfrak{T}_0$.

Theorem 6.5 (Knuth 1970). *The equivalence relation \equiv_{tab} coincides with the plactic congruence \equiv_{plc} which is the Knuth congruence \equiv_{knu} . In particular, each plactic class contains exactly one tableau.*

In our notation, the theorem reads as

$$\mathfrak{T}_w = [w]_{\text{plc}} (= [w]_{\text{knu}}), \quad \text{for every } w \in \mathcal{A}_n,$$

and hence we may alternate between these notations. Therefore Tab_n can be realized as an algebraic structure, equipped with a multiplicative operation.

Remark 6.6. *Set theoretically, Tab_n bijectively corresponds to the plactic monoid \mathcal{P}_n , thus it can be considered as a monoid whose operation is now induced from Algorithm 6.3, or equivalently from (6.1). That is, the product of tableaux $\mathfrak{T}_u \cdot \mathfrak{T}_v$, with say $\mathfrak{T}_v = y_1 \cdots y_m$, is defined by the sequential insertion (by reverse row ordering) of the letters y_1, \dots, y_m of \mathfrak{T}_v to \mathfrak{T}_u .*

In order to extract numerical invariants from the combinatorial structure of semi-standard tableaux we define the following functions.

(a) The function

$$\#_{a_\ell}^{(i)} : \text{Tab}_n \longrightarrow \mathbb{N}_0, \quad \ell, i = 1, \dots, n, \quad (6.2)$$

counts the occurrences of the letter $a_\ell \in \mathcal{A}_n^+$ in the i 'th row of a tableau $\mathfrak{T} \in \text{Tab}_n$.

(b) The function

$$\mathfrak{h}_J : \text{Tab}_n \longrightarrow \mathbb{N}_0, \quad J \subset \{1, \dots, n\}, \quad J \neq \emptyset, \quad (6.3)$$

assigns a tableau with length of its longest nondecreasing subword that involves only letters $a_\ell \in \mathcal{B}_J$ from the sub-alphabet $\mathcal{B}_J \subseteq \mathcal{A}_n$.

Our main interested is in convex sub-alphabet $\mathcal{A}_J \subseteq_{\text{cx}} \mathcal{A}_n$, i.e., $J = \{i, i+1, \dots, j\}$, with $i \leq j$, which we denote as $[a_i : a_j]$ or $[i : j]$, for short.

When $i = j = \ell$, the function $\mathfrak{h}_{[\ell:\ell]}(\mathfrak{T})$ gives the number of occurrences of the letter a_ℓ in the tableau \mathfrak{T} , and thus

$$\mathfrak{h}_{[\ell:\ell]}(\mathfrak{T}) = \sum_{t=1}^n \#_{a_\ell}^{(t)}(\mathfrak{T}), \quad \text{for every } \ell = 1, \dots, n.$$

We write $\mathfrak{T}_{[i:j]}$ for the restriction of the tableau \mathfrak{T} to rows i, \dots, j .

Remark 6.7. *The equivalence relation \equiv_{clk} on tableaux \mathfrak{T}_u and \mathfrak{T}_v in Tab_n (cf. (5.2)) reads in terms of (6.3) as*

$$\mathfrak{T}_u \equiv_{\text{clk}} \mathfrak{T}_v \Leftrightarrow \mathfrak{h}_J(\mathfrak{T}_u) = \mathfrak{h}_J(\mathfrak{T}_v) \quad \text{for every } J \subseteq_{\text{cx}} \{1, \dots, n\}.$$

(Note that if $\mathfrak{T}_u \equiv_{\text{clk}} \mathfrak{T}_v$, then \mathfrak{T}_u and \mathfrak{T}_v must contains exactly the same number of each letter.)

We start with our running example (appeared also in Remark 5.12) that points out special pathologies along our exposition.

Example 6.8. The words $u = c b^2 c^2 a^2 b^2 c$ and $v = c^2 b^2 c a^2 b^2 c$ have the following tableau realizations:

$$u = c b^2 c^2 a^2 b^2 c \mapsto \mathfrak{T}_u = \begin{array}{|c|c|c|c|c|} \hline c & & & & \\ \hline b & b & c & c & \\ \hline a & a & b & b & c \\ \hline \end{array},$$

and

$$v = c^2 b^2 c a^2 b^2 c \mapsto \mathfrak{T}_v = \begin{array}{|c|c|c|c|c|} \hline c & c & & & \\ \hline b & b & c & & \\ \hline a & a & b & b & c \\ \hline \end{array}.$$

It is easy to check that $u \equiv_{\text{clk}} v$, where $\mathfrak{l}_{[b:c]}(\mathfrak{T}_u) = \mathfrak{l}_{[b:c]}(\mathfrak{T}_v) = 5$, but $u \not\equiv_{\text{tab}} v$.

Hence, from this example, we learn that \equiv_{clk} does not imply \equiv_{tab} (or equivalently \equiv_{knu}), but the converse holds.

Proposition 6.9. If $u \equiv_{\text{knu}} v$, or equivalently $u \equiv_{\text{tab}} v$, then $u \equiv_{\text{clk}} v$, for any $u, v \in \mathcal{A}_n^*$. Thus $u \equiv_{\text{clk}} v$ iff $\mathfrak{T}_u \equiv_{\text{clk}} \mathfrak{T}_v$.

Proof. By Theorem 6.5, $u \equiv_{\text{knu}} v$ iff $u \equiv_{\text{tab}} v$ iff $\mathfrak{T}_u = \mathfrak{T}_v$, then it follows from Remark 6.7 that $u \equiv_{\text{clk}} v$. \square

Corollary 6.10. The map

$$\bar{\partial} : \mathcal{P}_n \longrightarrow \mathcal{K}_n, \quad [w]_{\text{plc}} \longmapsto [w]_{\text{clk}},$$

is a surjective homomorphism.

Proof. Both \mathcal{P}_n and \mathcal{K}_n admit the Knuth relations (KNT), cf. Corollary 5.7, where \equiv_{knu} implies \equiv_{clk} by Proposition 6.1. Thus $\bar{\partial}$ is a well defined monoid homomorphism. Surjectivity is clear. \square

Accordingly the diagram

$$\begin{array}{ccc} \mathcal{A}_n^* & \xrightarrow{\quad} & \mathcal{P}_n := \mathcal{A}_n^* / \equiv_{\text{knu}} \\ & \searrow & \downarrow \bar{\partial} \\ & & \mathcal{K}_n := \mathcal{A}_n^* / \equiv_{\text{clk}} \end{array}$$

commutes.

6.2. Configuration tableaux.

We introduce a new class of tableaux, carrying nonnegative integers. These tableaux record lengths of tracks in semi-standard tableaux as weights, and are used as axillary tableaux, playing a major role in this paper. The cells' indexing in these tableaux are arranged in a way that each diagonal corresponds to a letter, as explained below. As will be seen later, this indexing system makes sense in the passage to matrices which establish linear representations of Young tableaux.

Definition 6.11. An n -**configuration tableau** $\mathfrak{C} = (\lambda_{i,j})$ is an isosceles tableau with n rows and n columns of fixed shape ($n > n-1 > \dots > 1$) whose cells are indexed as

$$\mathfrak{C} = \begin{array}{|c|c|c|c|c|c|} \hline \lambda_{n,n} & & & & & \\ \hline \lambda_{n-1,n-1} & \lambda_{n-1,n} & & & & \\ \hline \vdots & & \ddots & & & \\ \hline \lambda_{3,3} & & & \lambda_{3,n} & & \\ \hline \lambda_{2,2} & \lambda_{2,3} & & \lambda_{2,n-1} & \lambda_{2,n} & \\ \hline \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \cdots & \lambda_{1,n-1} & \lambda_{1,n} \\ \hline \end{array},$$

the rows are enumerated from bottom to top and columns from left to right. The index i of $\lambda_{i,j}$ stands for i 'th row, while j refers to the j 'th diagonal (enumerated for bottom left).

The content of cells are nonnegative integers $\lambda_{i,j}$ satisfying the following **configuration laws** for all $i \leq j \leq n$

$$\sum_{t=0}^k \lambda_{j,j+t} \leq \sum_{t=0}^k \lambda_{i,i+t}, \quad \text{for every } k \leq n-j. \quad (6.4)$$

The **null configuration tableau**, denoted \mathfrak{C}_0 , is the n -configuration tableau with $\lambda_{i,j} = 0$ for all i, j . We denote the set of all n -configuration tableaux by \mathbf{CTab}_n .

The ℓ 'th **diagonal** of \mathfrak{C} , for fixed ℓ , is the collection of cells $\lambda_{t,\ell}$, $t = 1, \dots, \ell$. We define

$$\tau_\ell(\mathfrak{C}) := \sum_{t=1}^{\ell} \lambda_{t,\ell}, \quad \ell = 1, \dots, n, \quad (6.5)$$

which we call the ℓ 'th **trace** of \mathfrak{C} .

An n -configuration tableau is called **standard** if $\tau_\ell = 1$ for every $\ell = 1, \dots, n$. The set of all standard n -configuration tableaux is denoted by \mathbf{SCTab}_n .

Literally, the configuration laws (6.4) assert that the sum of cells' values of a left adjacent subrow of \mathfrak{r}_i is greater or equal than the sum of cells' values of any left adjacent subrow of the same length placed above \mathfrak{r}_i .

Note that *configuration tableaux by themselves need not be semi-standard* as they may contain 0-valued cells. Nevertheless, as proved later, they bijectively correspond to semi-standard tableaux. The fact that all n -configuration tableaux have the same fixed shape, together with a canonical indexing system of cells, permits uniform formal references, and provides a suitable notation for walks between cells.

Given an n -configuration tableau $\mathfrak{C} = (\lambda_{i,j})$, we write $\gamma_{i,j}$ for a proper walk from cell $\lambda_{i,i}$ to cell $\lambda_{1,j}$; $\Gamma_{i,j}$ denotes the set of all such proper walks $\gamma_{i,j}$. Due to the cells' indexing of configuration tableaux, since $\gamma_{i,j}$ is proper,

- $\gamma_{i,j}$ is a descending walk iff $i \leq j$,
- $\gamma_{i,j}$ is a falling walk iff $i \geq j$.

The **weight** $w(\gamma_{i,j})$ of the walk $\gamma_{i,j}$ is the sum of its cells' values, i.e.,

$$w(\gamma_{i,j}) := \sum_{\lambda_{s,t} \in \gamma_{i,j}} \lambda_{s,t}.$$

We write $\overline{w}_{i,j}(\mathfrak{C})$ for the maximal weight over all walks $\gamma_{i,j} \in \Gamma_{i,j}$, that is

$$\overline{w}_{i,j}(\mathfrak{C}) = \bigvee_{\gamma_{i,j} \in \Gamma_{i,j}} w(\gamma_{i,j}), \quad (i \leq j), \quad (6.6)$$

which determines the function

$$\overline{w}_{i,j} : \mathbf{CTab} \longrightarrow \mathbb{N}_0, \quad i, j = 1, \dots, n. \quad (6.7)$$

We write $\underline{w}_{i,j}(\mathfrak{C})$ for the minimal weight over all walks $\gamma_{i,j} \in \Gamma_{i,j}$, that is

$$\underline{w}_{i,j}(\mathfrak{C}) = \bigwedge_{\gamma_{i,j} \in \Gamma_{i,j}} w(\gamma_{i,j}), \quad (i \geq j). \quad (6.8)$$

When $i = j$, the set $\Gamma_{i,i}$ consists of exactly one proper walk $\gamma_{i,i}$ which is both a descending walk and a falling walk, and thus (cf. (6.5))

$$\overline{w}_{i,i}(\mathfrak{C}) = \underline{w}_{i,i}(\mathfrak{C}) = w(\gamma_{i,i}) = \tau_i$$

for any $i = 1, \dots, n$. On the other hand, for fixed $i = 1$, the weight

$$w(\gamma_{1,j}) = \overline{w}_{1,j}(\mathfrak{C}) = \sum_{t=1}^j \lambda_{1,t} \quad (6.9)$$

is determined by a unique (bottom) walk, which precisely encodes the first row \mathfrak{r}_1 of \mathfrak{C} . Hence, the weights of the walks $\gamma_{1,j}$ record the full data on \mathfrak{r}_1 .

Let $\Theta_{i,j}(\mathfrak{C})$ be the set of all horizontal covers $\theta_{i,j}$ of columns $1, \dots, j$ by rows $1, \dots, i$ with $i \geq j$, cf. §6.1. Namely, $\theta_{i,j}$ is a collection of cells $\lambda_{i_1,1}, \dots, \lambda_{i_j,j}$ in $\mathfrak{C} := (\lambda_{i,j})$ such that $i_1 \geq i_2 \geq \dots \geq i_j$ with $i_t \in \{1, \dots, j\}$; in particular $\theta_{i,i} = \lambda_{i,i}$ for each $i = 1, \dots, n$.

The **weight** $w(\theta_{i,j})$ of a cover $\theta_{i,j}$ is the sum of its cells' values, i.e.,

$$w(\theta_{i,j}) := \sum_{\lambda_{s,t} \in \theta_{i,j}} \lambda_{s,t}.$$

Using these weights we introduce a new function on n -configuration tableaux:

$$\eta_{i,j} : \mathbf{CTab} \longrightarrow \mathbb{N}_0, \quad i, j = 1, \dots, n, \quad (6.10)$$

defined for $i \geq j$ as

$$\eta_{i,j}(\mathfrak{C}) = \bigwedge_{\theta_{i,j} \in \Theta_{ij}} w(\theta_{i,j}), \quad (6.11)$$

and $\eta_{i,j}(\mathfrak{C}) := 0$ whenever $i < j$. Explicitly, in terms of subsequences, the value of the function $\eta_{i,j}$ can be computed directly as

$$\eta_{i,j}(\mathfrak{C}) = \bigwedge_{S_j \subseteq [1:i]} \sum_{\substack{t=1, \\ s_t \in S_j}}^j \lambda_{t,s_t}, \quad i \geq j.$$

In particular, for $i = j$ we have $\eta_{i,i}(\mathfrak{C}) = \tau_i$, while by the configuration law (6.4) for every $i = 1, \dots, n$ we obtain that

$$\eta_{i,1}(\mathfrak{C}) = \lambda_{i,i}.$$

So, we see that $\eta_{i,1}(\mathfrak{C})$, $i = 1, \dots, n$, records explicitly the full data on the first column \mathfrak{c}_1 of \mathfrak{C} .

The fixed structure of configuration tableaux enables a straightforward execution of operations that resize the tableaux – shrink or extend them.

Remark 6.12. *The deletion of the bottom row or the n 'th diagonal of an n -configuration tableau results in a new proper $(n-1)$ -configuration tableau. This does not hold for the deletion of the left column, as seen in Example 6.14 below.*

On the other hand, one can enlarge the size of a tableaux through tableau injections, fully preserving its content. The possible injections are determined by column mappings, depending on the size of the target tableau, while row mappings are always one-to-one.

Remark 6.13. *Among the possible injections of configuration tableaux of different sizes, we are interested in the **right injection**, that is the map*

$$\text{Inj}_r : \text{CTab}_m \longrightarrow \text{CTab}_n, \quad m \leq n,$$

that embeds an m -configuration tableau $\mathfrak{C} \in \text{CTab}_m$ in the n -configuration tableau whose $m \times m$ right part is identically \mathfrak{C} and all its cells in the left columns $1, \dots, n-m$ are of value 0.

6.3. Semi-standard tableaux vs. configuration tableaux.

Every semi-standard tableau $\mathfrak{T} \in \text{Tab}_n$ is associated to an n -configuration tableau $\mathfrak{C} \in \text{CTab}_n$ by the map

$$\mathcal{I}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n, \quad (6.12)$$

defined as

$$\mathcal{I}_{\text{ctab}} : \mathfrak{T} \longmapsto \mathfrak{C}, \quad \text{where } \lambda_{i,\ell} := \#_{a_\ell}^{(i)}(\mathfrak{T}), \quad (6.13)$$

i.e., the cell $\lambda_{i,\ell}$ of $\mathfrak{C} = (\lambda_{i,j})$ is assigned with the occurrence number of the letter a_ℓ in the row \mathfrak{r}_i of \mathfrak{T} , and $\mathfrak{T}_0 \longmapsto \mathfrak{C}_0$. By this setting the letter a_ℓ in \mathfrak{T} corresponds to the ℓ 'th diagonal of \mathfrak{C} , enumerated starting from bottom left. Accordingly, the ℓ -trace τ_ℓ of \mathfrak{C} , cf. (6.5), gives the total occurrences of the letter a_ℓ in \mathfrak{T} , which is read off from the ℓ 'th diagonal of \mathfrak{C} .

Example 6.14. *Let $w = a_4 a_3 a_4 a_2 a_2 a_4 a_4 a_1 a_1 a_1 a_3 a_3 a_4$ be a standard tableau for which the map $\mathcal{I}_{\text{ctab}} : \mathfrak{T}_w \longmapsto \mathfrak{C}$ is as follows:*

$$\mathfrak{T}_w = \begin{array}{|c|c|c|c|c|c|} \hline a_4 & & & & & \\ \hline a_3 & a_4 & & & & \\ \hline a_2 & a_2 & a_4 & a_4 & & \\ \hline a_1 & a_1 & a_1 & a_3 & a_3 & a_4 \\ \hline \end{array} \longrightarrow \mathfrak{C} = \begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline 1 & 1 & & \\ \hline 2 & 0 & 2 & \\ \hline 3 & 0 & 2 & 1 \\ \hline \end{array}$$

$\swarrow \quad \swarrow \quad \swarrow \quad \swarrow$
 $a_1 \ a_2 \ a_3 \ a_4$

The traces of \mathfrak{C} are then $\tau_1(\mathfrak{C}) = 3$, $\tau_2(\mathfrak{C}) = 2$, $\tau_3(\mathfrak{C}) = 3$, and $\tau_4(\mathfrak{C}) = 5$.

This example also shows that tableaux are not necessarily semi-standard, e.g., see the second column that is not (strictly) increasing.

Remark 6.15. *For every semi-standard tableau $\mathfrak{T} \in \text{Tab}_n$ assigning (6.13) admits the configuration laws (6.4) as for every $i < j$ the j 'th row \mathfrak{r}_j of \mathfrak{T} dominates its i 'th row \mathfrak{r}_i .*

An important property of the map (6.12) is that it also precisely records the shape of semi-standard tableaux. That is, the shape $(\lambda_1, \dots, \lambda_m)$ of a tableau \mathfrak{T} is encoded in its image $\mathcal{T}_{\text{ctab}}(\mathfrak{T})$ as the sums of cells' values of the rows:

$$\lambda_i = \sum_{t=i}^n \lambda_{i,t}.$$

In general, an n -configuration tableau records all the information that its pre-image semi-standard tableau carries, in a numerical way.

Theorem 6.16. *The map $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n$ in (6.12) is bijective.*

Proof. Let $\mathbf{r}_i = \alpha_{\ell_1}^{q_1} \alpha_{\ell_2}^{q_2} \cdots \alpha_{\ell_m}^{q_m}$, where $\ell_1, \dots, \ell_m \in L \subseteq \{i, \dots, n\}$ and $q_t \in \mathbb{N}$, be the i 'th row of the tableau $\mathfrak{T} \in \text{Tab}(\mathcal{A}_n)$. Then \mathbf{r}_i can be rewritten uniquely as $\alpha_i^{q_i} \alpha_{i+1}^{q_{i+1}} \cdots \alpha_n^{q_n}$ with $q_t \in \mathbb{N}_0$ for every $\ell_t \in \{i, \dots, m\}$ and $q_t = 0$ when $t \in \{i, \dots, n\} \setminus L$. Thus, the row mapping

$$\mathcal{T}_{\text{ctab}}|_{\mathbf{r}_i} : \alpha_i^{q_i} \alpha_{i+1}^{q_{i+1}} \cdots \alpha_n^{q_n} \longmapsto q_i \ q_{i+1} \ \cdots \ q_n \quad (*)$$

is injective, i.e., $\lambda_{i,t} = q_t$, $t = i, \dots, n$. As this holds for the restriction $\mathcal{T}_{\text{ctab}}|_{\mathbf{r}_i}$ of $\mathcal{T}_{\text{ctab}}$ to any row \mathbf{r}_i , and $\mathcal{T}_{\text{ctab}}$ maps row-to-row, we deduce that $\mathcal{T}_{\text{ctab}}$ is injective over the whole tableau (and also respects the configuration laws (6.4) by Remark 6.15).

To see that $\mathcal{T}_{\text{ctab}}$ is surjective, use (*) to reproduce each row \mathbf{r}_i of \mathfrak{T} from the i 'th row of \mathfrak{C} , which together, due to the configuration laws (6.4), obey the dominance relations in semi-standard tableaux. \square

The next conclusions are now immediate.

Corollary 6.17. *The restriction*

$$\mathcal{T}_{\text{stab}} : \text{STab}_n \longrightarrow \text{SCTab}_n \quad (6.14)$$

of $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n$ to standard tableaux is bijective.

In view of Theorem 6.16, and (6.1), we define the map

$$\text{ctab} : \mathcal{A}_n^+ \longrightarrow \text{CTab}_n, \quad w \longmapsto \mathcal{T}_{\text{ctab}}(\text{tab}(w)),$$

and write

$$\mathfrak{C}_w := \text{ctab}(w)$$

for short. For the empty word $e \in \mathcal{A}_n^*$, we formally set $\mathfrak{C}_e := \mathfrak{C}_0$ to be the null configuration tableau \mathfrak{C}_0 (Definition 6.11).

Corollary 6.18. *The map*

$$\mathcal{P}_{\text{ctab}} : \mathcal{P}_n := \mathcal{A}_n^* / \equiv_{\text{plc}} \longrightarrow \text{CTab}_n, \quad [w]_{\text{plc}} \longmapsto \mathfrak{C}_w,$$

is bijective, and $u \equiv_{\text{plc}} v$ iff $\mathfrak{C}_u = \mathfrak{C}_v$, for any $u, v, w \in \mathcal{A}_n^$.*

Proof. Compose Theorem 6.16 with Theorem 6.5. \square

Similar to Remark 6.6, by the latter corollary, the map $\mathcal{P}_{\text{ctab}} : \mathcal{P}_n \longrightarrow \text{CTab}_n$ can be realized as a monoid homomorphism, where CTab_n is a monoid whose operation (insertion by concatenating) is induced by the next algorithm. This view is compatible with the monoid structure of Tab_n via the bijection $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n$ – reads now a tableau isomorphism.

Algorithm 6.19 (Encoding Algorithm). *To encode a word $w \in \mathcal{A}_n^+$ in an n -configuration tableau \mathfrak{C} , start with an empty n -configuration tableau $\mathfrak{C} := \mathfrak{C}_e$ and perform the following letter by letter.*

To encode the letter $a_\ell \in \mathcal{A}_n$ in \mathfrak{C} , start with the first row $i = 1$

- *increment $\lambda_{i,\ell}$ by 1, i.e., $\lambda_{i,\ell} \leftarrow \lambda_{i,\ell} + 1$;*
- *decrement λ_{i,k_i} by 1, where $k_i > \ell$ is the minimal index with $\lambda_{i,k_i} > 0$.*

Repeat this same procedure to insert a_{k_i} to row $i + 1$ of \mathfrak{C} , as long as $i < n$.

This encoding of a word w in \mathfrak{C} is denoted by $w \mapsto \mathfrak{C}$, resulting in the configuration tableau \mathfrak{C}_w .

By this algorithm we see that the restriction of $\mathcal{A}_{[1:n]}^+$ to $\mathcal{A}_{[\ell:n]}^+$, i.e., to words over the convex sub-alphabet $\mathcal{A}_{[\ell:n]} := \{a_\ell, \dots, a_n\} \subseteq_{\text{cx}} \{a_1, \dots, a_n\}$, reads in terms of configuration tableaux as the image of right injection $\text{Inj}_r : \text{CTab}_{n-\ell+1} \hookrightarrow \text{CTab}_n$, cf. Remark 6.13.

Due to the special structure of configuration tableaux, provided by the configuration laws (6.4), we obtain the next important property.

Lemma 6.24. *Suppose $\mathfrak{C}_w = \text{ctab}(w)$ is a standard n -configuration tableau in SCTab_n with nonzero cells $\lambda_{i,k} = 1$ and $\lambda_{j,k+1} = 1$, where $i < j$. Then, for the nonzero cells $\lambda'_{i',k}$ and $\lambda'_{j',k+1}$ of $\mathfrak{C}'_w = \text{ctab}(\overline{w})$, we have $j' \leq i'$.*

Proof. Since $j > i$ where $\lambda_{i,k} = 1$ and $\lambda_{j,k+1} = 1$, the translation of $\mathfrak{C}_w = \text{ctab}(w)$ to $\mathfrak{T}_w = \text{tab}(w)$ by Theorem 6.16 means that in \mathfrak{T}_w the letter a_{k+1} is on the left to a_k , which implies that a_{k+1} is on the right to a_k in $\mathfrak{T}'_w = \text{tab}(\overline{w})$ by Proposition 6.22. Then, the one-to-one correspondence $\mathcal{T}_{\text{ctab}} : \mathfrak{T}'_w \mapsto \mathfrak{C}'_w$, by Theorem 6.16, gives the required. \square

Now we are first able to directly tied the underlying congruence \equiv_{clk} of the cloaktic monoid (Definition 5.1) to tableau equivalence \equiv_{tab} in the class of standard tableaux.

Theorem 6.25. *Let $\mathfrak{T}_u = \text{tab}(u)$ and $\mathfrak{T}_v = \text{tab}(v)$ be standard tableaux in STab_n , with $u, v \in \mathcal{A}_n^+$, then*

$$\mathfrak{T}_u = \mathfrak{T}_v \quad (\Leftrightarrow \mathfrak{T}'_u = \mathfrak{T}'_v) \quad \Leftrightarrow \quad u \equiv_{\text{clk}} v \quad \text{and} \quad \overline{u} \equiv_{\text{clk}} \overline{v},$$

where $\mathfrak{T}'_u = \text{tab}(\overline{u})$ and $\mathfrak{T}'_v = \text{tab}(\overline{v})$.

Proof. Recall that by definition $u \equiv_{\text{tab}} v$ iff $\mathfrak{T}_u = \mathfrak{T}_v$. The part $(\Leftrightarrow \mathfrak{T}'_u = \mathfrak{T}'_v)$ has already been proven in Proposition 6.22, which also asserts that $u \equiv_{\text{clk}} v$ iff $\mathfrak{T}_u \equiv_{\text{clk}} \mathfrak{T}_v$. We prove the rest.

(\Rightarrow) : Immediate since \equiv_{tab} implies \equiv_{clk} , by Proposition 6.9.

(\Leftarrow) : Let $\mathfrak{C}_u = \text{ctab}(u)$, $\mathfrak{C}_v = \text{ctab}(v)$, $\mathfrak{C}'_u = \text{ctab}(\overline{u})$, and $\mathfrak{C}'_v = \text{ctab}(\overline{v})$, and denote their cells respectively by $[\lambda_u]_{i,j}$, $[\lambda_v]_{i,j}$, $[\lambda'_u]_{i,j}$, and $[\lambda'_v]_{i,j}$. Assume that $\mathfrak{T}_u \neq \mathfrak{T}_v$, and thus $\mathfrak{T}'_u \neq \mathfrak{T}'_v$ by Proposition 6.22, implying that $\mathfrak{C}_u \neq \mathfrak{C}_v$ and $\mathfrak{C}'_u \neq \mathfrak{C}'_v$, by Corollary 6.17. Recall that each diagonal of a standard n -configuration tableau has a unique nonzero cell of value 1.

Proof by induction on n . The cases of $n = 1$ and $n = 2$ are clear. Assuming the implication holds for $\mathcal{A}_{n-1} \subseteq_{\text{cx}} \mathcal{A}_n$, the only change happens by including the extra letter a_n is at the n 'th diagonal of standard n -configuration tableaux. Then, by the induction assumption, \mathfrak{C}_u and \mathfrak{C}_v have the same $(n-1)$ 'th diagonal whose nonzero cell is say at $\lambda_{k,n-1} = 1$.

Suppose that $[\lambda_u]_{i,n} = 1$ and $[\lambda_v]_{j,n} = 1$, say for $j > i$, and assume that $u \equiv_{\text{clk}} v$ (or equivalently $\mathfrak{T}_u \equiv_{\text{clk}} \mathfrak{T}_v$), which in terms of the function (6.3) implies that the equality $\mathfrak{h}_{[s:t]}(\mathfrak{T}_u) = \mathfrak{h}_{[s:t]}(\mathfrak{T}_v)$ holds for all $1 \leq s \leq t \leq n$. We deduce that $j > i > k$, since otherwise we would get $\mathfrak{h}_{[i:n]}(\mathfrak{T}_u) > \mathfrak{h}_{[i:n]}(\mathfrak{T}_v)$.

Similarly, consider the n -configuration tableaux \mathfrak{C}'_u and \mathfrak{C}'_v of \overline{u} and \overline{v} respectively, and let $[\lambda'_u]_{k',n-1}$, $[\lambda'_u]_{i',n}$, and $[\lambda'_v]_{j',n}$ be respectively the nonzero cells of \mathfrak{C}'_u and \mathfrak{C}'_v , say with $i' > j'$. Then, assuming that $\overline{u} \equiv_{\text{clk}} \overline{v}$, by the same argument as above we obtain $i' > j' > k'$, since otherwise we would have $\mathfrak{h}_{[j':n]}(\mathfrak{C}'_v) > \mathfrak{h}_{[j':n]}(\mathfrak{C}'_u)$. But, since $j > i > k$, by Lemma 6.24 we should have $j' \leq k'$ and $i' \leq k' - 1$ — a contradiction. \square

7. REPRESENTATIONS OF TABLEAUX AND OF THE PLACTIC MONOID

In this section we utilize the tropical representations of the cloaktic monoid (§5.1) and the co-cloaktic monoid (§5.3), which essentially record lengths of longest subwords over convex sub-alphabets, to construct linear representations of semi-standard tableaux. Linear representations of the plactic monoid PLC_n are then follow from the correspondence between the elements of PLC_n and n -configuration tableaux (Corollary 6.18). The latter correspond uniquely to semi-standard tableaux via the map $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \xrightarrow{\sim} \text{CTab}_n$ (Theorem 6.16). Our next step is to establish the two maps

$$\begin{aligned} \mathcal{E}_{\text{mat}} : \text{CTab}_n &\longrightarrow \mathfrak{A}_n^\times, \\ \mathcal{E}_{\text{mat}}^{\text{co}} : \text{CTab}_n &\longrightarrow {}^\diamond \mathfrak{A}_n^\times, \end{aligned}$$

from n -configuration tableaux to tropical matrices. Then their compositions with the bijection $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n$ provide the maps

$$\begin{aligned} \mathcal{T}_{\text{mat}} &:= \mathcal{E}_{\text{mat}} \circ \mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow \mathfrak{A}_n^\times, \\ \mathcal{T}_{\text{mat}}^{\text{co}} &:= \mathcal{E}_{\text{mat}}^{\text{co}} \circ \mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow {}^\diamond \mathfrak{A}_n^\times. \end{aligned}$$

At this point, the digraph realization of tropical matrices (cf. §3.2) is of major importance. To make our image matrices more comprehensible, in what follows, for simplicity, we set the formal variable κ in the troplactic matrix algebra \mathfrak{A}_n to have fixed value $\kappa := 1$, cf. (4.4).

The study in this section is accompanied with a collection of pathological examples that demonstrate the difficulties towards faithfully representing tableaux. Our development is supported by the use of

configuration tableaux that enables an easier analysis and helps to better understanding the combinatorial arguments. But before that, in order to complete these representations, we need another crucial competent.

7.1. The co-plactic monoid.

Co-mirroring of words (Definition 1.6) leads to the following monoid structure, drawn from the plactic monoid PLC_n (Definition 2.1). Recall that \equiv_{plc} is an additional notation for the underlying congruence \equiv_{knu} of PLC_n .

Definition 7.1. *The **co-plactic monoid** is the monoid ${}^{\text{co}}\mathcal{P}_n := {}^{\text{co}}\text{PLC}(\mathcal{A}_n)$ generated by a finite ordered set of elements $\mathcal{A}_n := \{a_1, \dots, a_n\}$, subject to the equivalence relation $\equiv_{\text{plc}}^{\text{co}}$, defined as*

$$u \equiv_{\text{plc}}^{\text{co}} v \iff {}^{\text{co}}\mathfrak{M}(v) \equiv_{\text{plc}} {}^{\text{co}}\mathfrak{M}(u).$$

Namely ${}^{\text{co}}\mathcal{P}_n := \mathcal{A}_n^* / \equiv_{\text{plc}}^{\text{co}}$. We say that ${}^{\text{co}}\text{PLC}_n$ is of **rank** n , and write ${}^{\text{co}}\text{PLC}_n$ for ${}^{\text{co}}\text{PLC}(\mathcal{A}_n)$ when \mathcal{A}_n is arbitrary.

In other words, the relation $\equiv_{\text{plc}}^{\text{co}}$ is satisfied if the words $a'_\ell := {}^{\text{co}}\mathfrak{M}_n(a_\ell)$ in \mathcal{A}_n^* , (lexicographically) ordered as $a'_1 < a'_2 < \dots < a'_n$ (cf. Remark 1.7), admit the Knuth relations (KNT). Note that here we initially consider only finitely generated monoid, to have the co-mirroring map well defined.

Theorem 7.2. *The co-mirrors (1.4) of the generators a_1, \dots, a_n of the plactic monoid \mathcal{P}_n ,*

$$a'_\ell := {}^{\text{co}}\mathfrak{M}(a_\ell), \quad \ell = 1, \dots, n,$$

ordered as $a'_1 < a'_2 < \dots < a'_n$, admit the Knuth relations (KNT) and thus the congruence \equiv_{plc} of \mathcal{P}_n implies the equivalence $\equiv_{\text{plc}}^{\text{co}}$.

Configuration tableaux correspond bijectively to elements of \mathcal{P}_n (Corollary 6.18) and are utilized to prove the theorem by a straightforward computation, heavily based on the Encoding Algorithm 6.19. The proof is performed by induction on n , where the induction step assumes the implication for $\{a_2, \dots, a_n\}$, which corresponds to assuming the implication for the upper $n-1$ rows of an n -configuration tableau. The proof is fairly technical, including several cases, and appears in its full details in Appendix A, together with some additional examples.

Recall from Remark 1.7 that $\mathcal{A}_n^{*[1]}$ is the finitely monoid generated by ${}^{\text{co}}\mathfrak{M}(a_1), \dots, {}^{\text{co}}\mathfrak{M}(a_n)$, and associated with the homomorphism $\partial_1 : \mathcal{A}_n^* \longrightarrow t\mathcal{A}_n^{*[1]}$.

Corollary 7.3. *The congruence \equiv_{plc} implies the equivalence $\equiv_{\text{plc}}^{\text{co}}$ (which by itself is then a congruence), the map*

$$\partial := \partial_1 : \mathcal{P}_n \xrightarrow{\sim} \mathcal{P}_n^{[1]}$$

is a monoid isomorphism, and the plactic monoid $\mathcal{P}_n := \mathcal{A}_n^ / \equiv_{\text{plc}}$ of rank n is also a co-plactic monoid ${}^{\text{co}}\mathcal{P}_n$ of rank n for which the map*

$$\tilde{\partial} : \mathcal{P}_n \twoheadrightarrow {}^{\text{co}}\mathcal{P}_n, \quad [w]_{\text{plc}} \mapsto [w]_{\text{cplc}} \quad (7.1)$$

is a surjective homomorphism.

Corollary 7.4. *The plactic monoid $\mathcal{P}_n = \langle a_1, \dots, a_n \rangle$ contains the inductive chain of plactic submonoids*

$$\mathcal{P}_n = \mathcal{P}_n^{[0]} \supset \mathcal{P}_n^{[1]} \supset \mathcal{P}_n^{[2]} \supset \dots$$

of rank n , generated by $a_1^{[i]} = {}^{\text{co}}\mathfrak{M}_n(a_1^{[i-1]})$, \dots , $a_n^{[i]} = {}^{\text{co}}\mathfrak{M}_n(a_n^{[i-1]})$. Furthermore, the maps

$$\partial_i : \mathcal{P}_n^{[i-1]} \xrightarrow{\sim} \mathcal{P}_n^{[i]}, \quad i = 1, 2, \dots,$$

cf. Remark 1.7, are monoid isomorphisms.

Remark 7.5. *The surjective homomorphism $\tilde{\partial} : \mathcal{P}_n \twoheadrightarrow \mathcal{K}_n$ (Corollary 6.10) induces the surjective homomorphism ${}^{\text{co}}\tilde{\partial} : \mathcal{P}_n \longrightarrow {}^{\text{co}}\mathcal{K}_n$ via the diagram*

$$\begin{array}{ccccc} \mathcal{P}_n & \xrightarrow{\tilde{\partial}} & \mathcal{K}_n & \xrightarrow{\sim} & \mathfrak{A}_n^\times \\ & \searrow {}^{\text{co}}\tilde{\partial} & & & \uparrow \\ \downarrow & & & & \downarrow \\ {}^{\text{co}}\mathcal{P}_n & \xrightarrow{\tilde{\partial}} & {}^{\text{co}}\mathcal{K}_n & \xrightarrow{\sim} & {}^\diamond\mathfrak{A}_n^\times \end{array} \quad (7.2)$$

where \mathcal{K}_n and ${}^{\circ}\mathcal{K}_n$ are linearly represented by \mathfrak{A}_n^\times and ${}^\diamond\mathfrak{A}_n^\times$, respectively.

7.2. Representations of configuration tableaux.

We start by defining our first map

$$\mathcal{C}_{\text{mat}} : \text{CTab}_n \longrightarrow \mathfrak{A}_n^\times, \quad (7.3)$$

that sends an n -configuration tableau $\mathfrak{C} = (\lambda_{i,j})$ to the matrix $U = (u_{i,j})$ determined by

$$u_{i,j} := \begin{cases} \overline{\omega}_{i,j}(\mathfrak{C}) & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases} \quad (7.4)$$

where $\overline{\omega}_{i,j}$ is given by (6.6) in §6.3. This map is then realized as a monoid homomorphism.

Lemma 7.6. *The matrix $W = \mathcal{C}_{\text{mat}}(\mathfrak{C})$ indeed belongs to \mathfrak{A}_n^\times , cf. (4.4), for any $\mathfrak{C} \in \text{CTab}_n$.*

Proof. $\mathcal{P}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n$ is bijective (Theorem 6.16), with $\mathfrak{b}_{[i:j]}(\mathfrak{T}_w) = \overline{\omega}_{i,j}(\mathfrak{C}_w)$ by Observation 6.20, while Tab_n and CTab_n are isomorphic (as monoids) to the plactic monoid \mathcal{P}_n (Remark 6.6 and Corollary 6.18, respectively). Furthermore, the map $\mathfrak{d} : \mathcal{P}_n \longrightarrow \mathcal{K}_n$, $[w]_{\text{plc}} \mapsto [w]_{\text{clk}}$, is a surjective monoid homomorphism (Corollary 6.10), while $\mathfrak{U} : \mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times$, $a_\ell \mapsto A_{(\ell)}$ for $\ell = 1, \dots, n$, is an isomorphism (Theorem 5.6). Thus we only need to show that \mathcal{C}_{mat} maps the n -configuration tableaux assigned to letters a_ℓ in \mathcal{A}_n to the generating matrices $A_{(\ell)}$ of \mathfrak{A}_n^\times , namely that $\mathcal{C}_{\text{mat}}(\mathfrak{C}_{a_\ell}) = A_{(\ell)}$ for each $\ell = 1, \dots, n$.

Drawing the n -configuration tableau of a letter a_ℓ , which is of the form

$$\mathfrak{C}_{a_\ell} = \text{ctab}(a_\ell) = \begin{array}{cccccccc} 0 & & & & & & & \\ 0 & 0 & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ 0 & & & & 0 & & & \\ \vdots & & & \dots & & \ddots & & \\ 0 & \dots & & & & \dots & 0 & \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 & \end{array}, \quad (7.5)$$

ℓ

by the definition of descending walks (§6.1) and their weights (6.6) we observe that

- (i) $\overline{\omega}_{i,j}(\mathfrak{C}_{a_\ell}) = 0$ for $i \leq j < \ell$;
- (ii) $\overline{\omega}_{i,j}(\mathfrak{C}_{a_\ell}) = 1$ for $i \leq \ell \leq j$;
- (iii) $\overline{\omega}_{i,j}(\mathfrak{C}_{a_\ell}) = 0$ for $\ell < i \leq j$.

Accordingly, the matrix $\mathcal{C}_{\text{mat}}(\mathfrak{C}_{a_\ell})$ defined by (7.4) is precisely the triangular matrix $A_{(\ell)} := F_{(\ell)} \vee E$ defined in (4.1) and (4.4), with $\kappa = 1$ and E as in (4.3). \square

By Remark 6.6 we conclude that:

Theorem 7.7. $\mathcal{C}_{\text{mat}} : \text{CTab}_n \longrightarrow \mathfrak{A}_n^\times$, given in (7.4), is a surjective map, realized as a monoid homomorphism.

Proof. We know that $\mathcal{P}_{\text{ctab}} : \mathcal{P}_n \longrightarrow \text{CTab}_n$ is a bijection (Corollary 6.18) realized as a monoid homomorphism, $\mathfrak{d} : \mathcal{P}_n \longrightarrow \mathcal{K}_n$ is a surjective monoid homomorphism (Corollary 6.10), and $\mathfrak{U} : \mathcal{K}_n \longrightarrow \mathfrak{A}_n^\times$ is an isomorphism (Theorem 5.6), thus the diagram

$$\begin{array}{ccc} \mathcal{P}_n & \xrightleftharpoons[\sim]{\mathcal{P}_{\text{ctab}}} & \text{CTab}_n \\ & \searrow \mathfrak{d} & \swarrow \mathcal{C}_{\text{mat}} \\ & \mathcal{K}_n & \xrightarrow[\sim]{\mathfrak{U}} \mathfrak{A}_n^\times \end{array}$$

commutes by Lemma 7.6. \square

Example 7.8. Applying the map \mathcal{C}_{mat} in (7.3) to the 3-configuration tableau \mathfrak{C}_w with w the word in Example 6.14, we have

$$\mathfrak{T}_w = \begin{array}{|c|c|c|c|c|c|c|} \hline a_4 & & & & & & \\ \hline a_3 & a_4 & & & & & \\ \hline a_2 & a_2 & a_4 & a_4 & & & \\ \hline a_1 & a_1 & a_1 & a_3 & a_3 & a_4 & \\ \hline \end{array} \mapsto \mathfrak{C}_w = \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline 1 & 1 & & & \\ \hline 2 & 0 & 2 & & \\ \hline 3 & 0 & 2 & 1 & \\ \hline \end{array} \mapsto \mathcal{C}_{\text{mat}}(\mathfrak{C}_w) = \begin{bmatrix} 3 & 3 & 5 & 6 \\ & 2 & 4 & 5 \\ & & 3 & 5 \\ & & & 5 \end{bmatrix},$$

where \mathfrak{T}_w is the Young tableau of w .

In terms of configuration tableaux, and the map $\mathcal{C}_{\text{mat}} : \text{CTab}_n \longrightarrow \mathfrak{A}_n^\times$, Example 5.10 now reads as follows.

Example 7.9. The representation $\mathcal{C}_{\text{mat}} : \text{CTab}_3 \longrightarrow \mathfrak{A}_3^\times$ of 3-configuration tableaux assigned to the three letter alphabet $\mathcal{A}_3 = \langle a, b, c \rangle$ is determined by the generators' mapping

$$\begin{aligned} \mathfrak{C}_a := \text{ctab}(a) &= \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 1 & 0 & 0 \\ \hline \end{array} \mapsto A = \begin{bmatrix} 1 & 1 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, & \mathfrak{C}_b := \text{ctab}(b) &= \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 0 & 1 & 0 \\ \hline \end{array} \mapsto B = \begin{bmatrix} 0 & 1 & 1 \\ & 1 & 1 \\ & & 0 \end{bmatrix}, \\ \mathfrak{C}_c := \text{ctab}(c) &= \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 0 & 0 & 1 \\ \hline \end{array} \mapsto C = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 1 \end{bmatrix}. \end{aligned}$$

The Knuth-equivalence (KNT) for triplets in \mathcal{A}_3 are described in CTab_3 and \mathfrak{A}_3^\times as

$$\text{tab}(acb) = \text{tab}(cab) = \begin{array}{|c|c|} \hline c & \\ \hline a & b \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 0 & 1 & & \\ \hline 1 & 1 & 0 & \\ \hline \end{array} \mapsto \begin{bmatrix} 1 & 2 & 2 \\ & 1 & 1 \\ & & 1 \end{bmatrix},$$

while

$$\text{tab}(bca) = \text{tab}(bca) = \begin{array}{|c|c|} \hline b & \\ \hline a & c \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 1 & 0 & & \\ \hline 1 & 0 & 1 & \\ \hline \end{array} \mapsto \begin{bmatrix} 1 & 1 & 2 \\ & 1 & 2 \\ & & 1 \end{bmatrix}.$$

For the increasing word $abc \in \mathcal{A}_3^+$ we get

$$\text{tab}(abc) = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 0 & 0 & & \\ \hline 1 & 1 & 1 & \\ \hline \end{array} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{bmatrix},$$

and for the decreasing word $cba \in \mathcal{A}_3^+$ we have

$$\text{tab}(cba) = \begin{array}{|c|c|} \hline c & \\ \hline b & \\ \hline a & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline 1 & 0 & & \\ \hline 1 & 0 & 0 & \\ \hline \end{array} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}.$$

For a generic 3-letter tableau $\mathfrak{T}_w := c^{k_3} b^{j_2} c^{k_2} a^{i_1} b^{j_1} c^{k_1}$, by a direct computation, we obtain

$$\mathfrak{C}_w = \begin{array}{|c|c|c|} \hline k_3 & & \\ \hline j_2 & k_2 & \\ \hline i_1 & j_1 & k_1 \\ \hline \end{array} \mapsto \mathcal{C}_{\text{mat}}(\mathfrak{C}_w) = \begin{bmatrix} i_1 & i_1 j_1 & i_1 j_1 k_1 \\ & j_1 j_2 & j_2(j_1 \vee k_2) \\ & & k_1 k_2 k_3 \end{bmatrix}.$$

Then taking k'_2, k''_2 such that $k'_2, k''_2 \leq j_2$ and $k'_2 k'_3 = k''_2 k''_3$, we can produce two different tableaux with a same image, what shows that in general the map \mathcal{C}_{mat} not injective.

Returning to our running example (cf. Example 6.8), it gives an explicit numerically example for the non-injectivity of the map $\mathcal{C}_{\text{mat}} : \text{Tab}_n \longrightarrow \mathfrak{A}_n^\times$.

Example 7.10. Taking the explicit words of Example 6.8, we have

$$u = c \ b^2 c^2 \ a^2 b^2 c \mapsto \mathfrak{C}_u = \begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline 2 & 2 & & \\ \hline 2 & 2 & 1 & \\ \hline \end{array} \mapsto A_u = \begin{bmatrix} 2 & 4 & 5 \\ & 4 & 5 \\ & & 4 \end{bmatrix},$$

where on the other also

$$v = c^2 \ b^2 c \ a^2 b^2 c \mapsto \mathfrak{C}_v = \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 2 & 1 & & \\ \hline 2 & 2 & 1 & \\ \hline \end{array} \mapsto A_v = \begin{bmatrix} 2 & 4 & 5 \\ & 4 & 5 \\ & & 4 \end{bmatrix}.$$

Hence $\mathcal{C}_{\text{mat}}(\mathfrak{C}_u) = \mathcal{C}_{\text{mat}}(\mathfrak{C}_v)$, but $u \not\equiv_{\text{tab}} v$, or equivalently $u \not\equiv_{\text{plc}} v$ by Theorem 6.5. Namely, the map $\mathcal{C}_{\text{mat}} : \text{CTab}_3 \rightarrow \mathfrak{A}_3^\times$ is not injective

Nevertheless, we do have a partial (row) injection:

Lemma 7.11. *The restriction*

$$\mathcal{C}_{\text{mat}}|_{\text{row}_1} : \text{row}_1(\mathfrak{C}) \longrightarrow \text{row}_1(W), \quad W = \mathcal{C}_{\text{mat}}(\mathfrak{C}), \quad (7.6)$$

of the map \mathcal{C}_{mat} in (7.3) to the bottom row is a bijective map.

Proof. By Theorem 6.16, $\text{row}_1(\mathfrak{C})$ corresponds uniquely to a nondecreasing word in \mathcal{A}_n^+ , which is mapped bijectively to $\text{row}_1(W)$ of $W \in \mathfrak{A}_n^\times$ by Lemma 5.9. \square

7.3. Co-representations of configuration tableaux.

We now turn to our second map

$$\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_n \longrightarrow {}^\diamond \mathfrak{A}_n^\times, \quad (7.7)$$

which we called a **co-representation**, that uses as its target the matrix co-monoid ${}^\diamond \mathfrak{A}_n^\times$, cf. (5.9). This map sends an n -configuration tableau $\mathfrak{C} \in \text{CTab}_n$ to the matrix $V = (v_{i,j})$ defined by

$$v_{i,j} := \begin{cases} -\eta_{i',j'}(\mathfrak{C}) & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases} \quad (7.8)$$

where $i' = n - i + 1$, $j' = n - j + 1$, and $\eta_{i',j'}$ is the function given by (6.11) in §6.3. The map $\mathcal{C}_{\text{mat}}^{\text{co}}$ is then realized as a monoid homomorphism. (Recall that, by Theorem 5.20, ${}^\diamond \mathfrak{A}_n^\times$ is isomorphic to the co-cloactic monoid ${}^{\text{co}}\mathcal{K}_n$.)

Lemma 7.12. *The matrix $V = \mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C})$ indeed belongs to ${}^\diamond \mathfrak{A}_n^\times$ for any $\mathfrak{C} \in \text{CTab}_n$.*

Proof. We follow similar arguments as in the proof of Lemma 7.6, with an additional step. First, $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \rightarrow \text{CTab}_n$ is a bijection (Theorem 6.16) with Tab_n and CTab_n isomorphic (as monoids) to the plactic monoid \mathcal{P}_n (Theorem 6.16, Remark 6.6, and Corollary 6.18, respectively). On the other hand, $\bar{\partial} : \mathcal{P}_n \rightarrow \mathcal{K}_n$, $[w]_{\text{plc}} \mapsto [w]_{\text{clk}}$, is a surjective homomorphism (Corollary 6.10), while $\bar{\cup} : \mathcal{K}_n \rightarrow \mathfrak{A}_n^\times$, $a_\ell \mapsto A_{(\ell)}$ is an isomorphism (Theorem 5.6). Finally we have the surjective homomorphism $\bar{\cap} : \mathfrak{A}_n^\times \rightarrow {}^\diamond \mathfrak{A}_n^\times$, $A_{(\ell)} \mapsto \check{A}_{(\ell)}$, cf. (5.12). Thus we only need to show that $\mathcal{C}_{\text{mat}}^{\text{co}}$ maps each n -configuration tableau assigned to the letter a_ℓ in \mathcal{A}_n to the matrix $A_{(\ell)}$ of \mathfrak{A}_n^\times , namely that $\mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_{a_\ell}) = \check{A}_{(\ell)}$.

Observing the n -configuration tableau \mathfrak{C}_{a_ℓ} assigned to the letter a_ℓ , see (7.5), by the definition of horizontal covers and (6.11) we obtain that:

- (i) $\eta_{i,j}(\mathfrak{C}) = 1$ for $i = j = \ell$;
- (ii) $\eta_{i,j}(\mathfrak{C}) = 0$ otherwise.

Then the image matrix $\mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_{a_\ell})$ defined by (7.8) is precisely the triangular matrix $\check{A}_{(\ell)} := F_{(\ell)} \vee E$ in (5.6), with $\kappa = 1$ and $\mathbb{1} = 0$, as usual. \square

Theorem 7.13. $\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_n \rightarrow {}^\diamond \mathfrak{A}_n^\times$, given by (7.8), is a surjective map, realized as a monoid homomorphism.

Proof. We know that $\mathcal{P}_{\text{ctab}} : \mathcal{P}_n \rightarrow \text{CTab}_n$ is a bijection (Corollary 6.18) realized as a monoid homomorphism, $\partial : \mathcal{P}_n \rightarrow {}^{\text{co}}\mathcal{P}_n$ is a surjective homomorphism (Corollary 7.3), ${}^{\text{co}}\bar{\partial} : {}^{\text{co}}\mathcal{P}_n \rightarrow {}^{\text{co}}\mathcal{K}_n$ is a surjective homomorphism induced from $\bar{\partial} : \mathcal{P}_n \rightarrow \mathcal{K}_n$ (Corollary 6.10), and $\Omega : \mathcal{K}_n \xrightarrow{\sim} {}^\diamond \mathfrak{A}_n^\times$ is an isomorphism (Theorem 5.20). Hence, by Lemma 7.12, the diagram

$$\begin{array}{ccc} \mathcal{P}_n & \xleftrightarrow[\sim]{\mathcal{P}_{\text{ctab}}} & \text{CTab}_n \\ \downarrow \partial & & \searrow \mathcal{C}_{\text{mat}}^{\text{co}} \\ {}^{\text{co}}\mathcal{P}_n & \xrightarrow{\bar{\partial}} & {}^{\text{co}}\mathcal{K}_n \xrightarrow[\sim]{\Omega} {}^\diamond \mathfrak{A}_n^\times \end{array}$$

commutes. \square

In comparison to the representation $\mathcal{C}_{\text{mat}} : \text{CTab}_3 \rightarrow \mathfrak{A}_3^\times$ in Example 7.9, for the co-representation $\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_3 \rightarrow {}^\diamond \mathfrak{A}_3^\times$ we have the following.

Example 7.14. The 3-configuration tableaux corresponding to the three letter alphabet $\mathcal{A}_3 = \langle a, b, c \rangle$ are co-represented by $\mathcal{C}_{\text{mat}}^{\text{co}}$ in ${}^\diamond\mathfrak{A}_3^\times$ by the matrices

$$\begin{aligned} \mathfrak{C}_a := \text{ctab}(a) &= \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 1 & 0 & 0 \\ \hline \end{array} \mapsto \check{A} = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & -1 \end{bmatrix}, & \mathfrak{C}_b := \text{ctab}(b) = \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 0 & 1 & 0 \\ \hline \end{array} \mapsto \check{B} = \begin{bmatrix} 0 & 0 & 0 \\ & -1 & 0 \\ & & 0 \end{bmatrix}, \\ \mathfrak{C}_c := \text{ctab}(c) &= \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 0 & 0 & 1 \\ \hline \end{array} \mapsto \check{C} = \begin{bmatrix} -1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}. \end{aligned}$$

The Knuth-equivalence (KNT) for triplets in \mathcal{A}_3 are described in ${}^\diamond\mathfrak{A}_3^\times$ as

$$\text{tab}(acb) = \text{tab}(cab) = \begin{array}{|c|c|} \hline c & \\ \hline a & b \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 1 & \\ \hline 1 & 1 & 0 \\ \hline \end{array} \mapsto \begin{bmatrix} -1 & -1 & 0 \\ & -1 & 0 \\ & & -1 \end{bmatrix},$$

while

$$\text{tab}(bca) = \text{tab}(bca) = \begin{array}{|c|c|} \hline b & \\ \hline a & c \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 1 & 0 & \\ \hline 1 & 0 & 1 \\ \hline \end{array} \mapsto \begin{bmatrix} -1 & 0 & 0 \\ & -1 & -1 \\ & & -1 \end{bmatrix}.$$

For the increasing word $abc \in \mathcal{A}_3^*$ we get

$$\text{tab}(abc) = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 1 & 1 & 1 \\ \hline \end{array} \mapsto \begin{bmatrix} -1 & 0 & 0 \\ & -1 & 0 \\ & & -1 \end{bmatrix},$$

and for the decreasing word $cba \in \mathcal{A}_3^*$ we have

$$\text{tab}(cba) = \begin{array}{|c|c|} \hline c & \\ \hline b & \\ \hline a & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 1 & 0 & \\ \hline 1 & 0 & 0 \\ \hline \end{array} \mapsto \begin{bmatrix} -1 & -1 & -1 \\ & -1 & -1 \\ & & -1 \end{bmatrix}.$$

But

$$\text{tab}(aca) = \text{tab}(caa) = \begin{array}{|c|c|} \hline c & \\ \hline a & a \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 1 & \\ \hline 2 & 0 & 0 \\ \hline \end{array} \mapsto \begin{bmatrix} -1 & 0 & 0 \\ & 0 & 0 \\ & & -2 \end{bmatrix},$$

where we also have

$$\text{tab}(aac) = \begin{array}{|c|c|c|} \hline a & a & c \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & 0 & \\ \hline 2 & 0 & 1 \\ \hline \end{array} \mapsto \begin{bmatrix} -1 & 0 & 0 \\ & 0 & 0 \\ & & -2 \end{bmatrix},$$

which shows that in general the co-representation $\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_n \longrightarrow {}^\diamond\mathfrak{A}_n^\times$ is not injective.

For a general 3-letter tableau $\mathfrak{T}_w := c^{k_3} b^{j_2} c^{k_2} a^{i_1} b^{j_1} c^{k_1}$, by a direct computation, we have

$$\mathfrak{C}_w = \begin{array}{|c|c|c|} \hline k_3 & & \\ \hline j_2 & k_2 & \\ \hline i_1 & j_1 & k_1 \\ \hline \end{array} \mapsto \check{A}_w = \begin{bmatrix} -(k_1 k_2 k_3) & -k_3(-j_1 \vee -k_2) & -k_3 \\ & -(j_1 j_2) & -j_2 \\ & & -i_1 \end{bmatrix},$$

showing that the values j_1, k_2 make the map $\mathcal{C}_{\text{mat}}^{\text{co}}$ non-injective. Recall that in Example 7.9, the cause for non-injectivity was j_2, k_2 .

Yet, in other situations, the use of co-representations $\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_n \longrightarrow {}^\diamond\mathfrak{A}_n^\times$ can be useful.

Example 7.15. Applying the co-representation $\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_3 \longrightarrow {}^\diamond\mathfrak{A}_3^\times$ to the 3-configuration tableaux in Example 7.10, we have

$$u = c b^2 c^2 a^2 b^2 c \longrightarrow \mathfrak{C}_u = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 2 & \\ \hline 2 & 2 & 1 \\ \hline \end{array} \longrightarrow \begin{bmatrix} -4 & -3 & -1 \\ & -4 & -2 \\ & & -2 \end{bmatrix},$$

where on the other

$$v = c^2 b^2 c a^2 b^2 c \longrightarrow \mathfrak{C}_v = \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 2 & 1 & \\ \hline 2 & 2 & 1 \\ \hline \end{array} \longrightarrow \begin{bmatrix} -4 & -3 & -2 \\ & -4 & -2 \\ & & -2 \end{bmatrix}.$$

Hence $\mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_u) \neq \mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_v)$, with $u \not\equiv_{\text{tab}} v$. Recall from Example 7.10 that for these tableaux we had $\mathcal{C}_{\text{mat}}(\mathfrak{C}_u) = \mathcal{C}_{\text{mat}}(\mathfrak{C}_v)$ which has shown that $\mathcal{C}_{\text{mat}} : \text{CTab}_n \longrightarrow \mathfrak{A}_n^\times$ is not injective.

Corollary 7.16. $u \equiv_{\text{knu}} v$ implies both $u \equiv_{\text{clk}} v$ and $u \equiv_{\text{clk}}^{\text{co}} v$.

Proof. The implication for the equivalence \equiv_{clk} has been already proven in Proposition 6.9, while that for the equivalence $\equiv_{\text{clk}}^{\text{co}}$ follows from Theorem 7.13. \square

7.4. Linear representations of the plactic monoid.

The representations of n -configuration tableaux as constructed previously in §7.2 and §7.3 lead directly to a representation of semi-standard tableaux, via the one-to-one correspondence $\mathcal{T}_{\text{ctab}} : \text{Tab}_n \longrightarrow \text{CTab}_n$ (Theorem 6.16), and thus to a linear representation of the plactic monoid (cf. Remark 6.6).

Theorem 7.17. *The map*

$$\begin{aligned} \wp_n : \mathcal{P}_n &\longrightarrow \mathfrak{A}_n^\times \times {}^\diamond \mathfrak{A}_n^\times, \\ \mathfrak{T}_w &\longmapsto (\mathcal{C}_{\text{mat}}(\mathfrak{C}_w), \mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_w)), \end{aligned} \quad (7.9)$$

is a surjective monoid homomorphism – a linear representation of the plactic monoid \mathcal{P}_n .

Proof. Compose the bijection $\mathcal{P}_{\text{ctab}} : \mathcal{P}_n \longrightarrow \text{CTab}_n$ (Corollary 6.18) separately with the two surjections $\mathcal{C}_{\text{mat}} : \text{CTab}_n \longrightarrow \mathfrak{A}_n^\times$ given by (7.4) (Theorem 7.7) and $\mathcal{C}_{\text{mat}}^{\text{co}} : \text{CTab}_n \longrightarrow {}^\diamond \mathfrak{A}_n^\times$ given by (7.8) (Theorem 7.13), realized as monoid homomorphisms by Remark 6.6. \square

Let $u = bdac$ and $v = dbac$, relating respectively the 4-configuration tableaux

$$\mathfrak{C}_u = \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 0 & 0 & & \\ \hline 1 & 0 & 1 & \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \neq \mathfrak{C}_v = \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 0 & 1 & & \\ \hline 1 & 0 & 0 & \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array},$$

or equivalently $\mathfrak{T}_u \neq \mathfrak{T}_v$, and therefore $u \not\equiv_{\text{plc}} v$. On the other hand, their image under the representation $\wp_4 : \mathcal{P}_4 \longrightarrow \mathfrak{A}_4^\times \times {}^\diamond \mathfrak{A}_4^\times$ is the same

$$\wp_4(u) = \wp_4(v) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -1 & 0 & 0 \\ & -1 & 0 & 0 \\ & & -1 & -1 \\ & & & -1 \end{bmatrix},$$

which shows that in general \wp_n is not injective for $n > 3$. Yet, this representation is faithful for the case of the plactic monoid \mathcal{P}_3 of rank 3.

Theorem 7.18. *The homomorphism $\wp_3 : \mathcal{P}_3 \longrightarrow \mathfrak{A}_3^\times \times {}^\diamond \mathfrak{A}_3^\times$ in (7.9) is an isomorphism – a faithful linear representation of the plactic monoid \mathcal{P}_3 .*

Proof. Proof by a straightforward computation. Suppose $\mathfrak{T}_w \in \text{Tab}_n$ is a tableau of the form

$$\mathfrak{T}_w := c^{k_3} b^{j_2} c^{k_2} a^{i_1} b^{j_1} c^{k_1}.$$

corresponding uniquely to the 3-configuration tableau $\mathfrak{C}_w = \text{ctab}(w)$, by Theorem 6.16. As in Theorem 7.17, define \wp_3 to have the generators' mapping

$$\mathfrak{C}_a \longmapsto (A, \check{A}), \quad \mathfrak{C}_b \longmapsto (B, \check{B}), \quad \mathfrak{C}_c \longmapsto (C, \check{C}),$$

where $A, B, C \in \mathfrak{A}_3^\times$ and $\check{A}, \check{B}, \check{C} \in {}^\diamond \mathfrak{A}_3^\times$ are described in Example 7.9 and 7.14, respectively.

For simplicity, write $i = i_1$, $j = j_1 j_2$, $k = k_1 k_2 k_3$. The direct computation of the matrix products

$$X = C^{k_3} B^{j_2} C^{k_2} A^{i_1} B^{j_1} C^{k_1}, \quad Y = \check{C}^{k_3} \check{B}^{j_2} \check{C}^{k_2} \check{A}^{i_1} \check{B}^{j_1} \check{C}^{k_1},$$

in $\text{TMat}(\mathbb{T})$ results in the correspondence

$$\mathfrak{C}_w = \begin{array}{|c|c|c|} \hline k_3 & & \\ \hline j_2 & k_2 & \\ \hline i_1 & j_1 & k_1 \\ \hline \end{array} \longmapsto X \times Y = \begin{bmatrix} i & i_1 j_1 & i_1 j_1 k_1 \\ & j & j_2(j_1 \vee k_2) \\ & & k \end{bmatrix} \times \begin{bmatrix} -k & -k_3(-j_1 \vee -k_2) & -k_3 \\ & -j & -j_2 \\ & & -i \end{bmatrix},$$

and we need to prove that \mathfrak{C}_w can be recover uniquely from the matrices $X = (x_{i,j})$ and $Y = (y_{i,j})$.

First i_1 , j_2 , and k_3 can be read off immediately from the right column of Y , while j_1 and k_1 are extracted recursively from the top row of X . So, it remains to compute j_2 and k_2 .

Since j_1 and j_2 have been already recovered, from $x_{2,3}$ we can figure out whether $k_2 \geq j_1$ or not. If $k_2 \geq j_1$, then $x_{2,3} = j_2 k_2$, which provides k_2 . Otherwise $k_2 < j_1$, which implies $-k_2 > -j_1$, and thus $y_{1,2} = (-k_2)(-k_3)$, which gives k_2 , as k_3 is already known. (Note that k_2 can also be computed as $(-k)/((-k_1)(-k_3))$.) \square

Corollary 7.19. *The plactic monoid \mathcal{P}_3 admits all the semigroup identities satisfied by the monoid $\text{TMat}_3(\mathbb{T})$ of triangular tropical matrices, in particular the semigroup identities (3.6) with $n = 3$*

$$\Pi_{(C,2,2)} : \quad yx^2y^2x \underline{x} yx^2y^2x = yx^2y^2x \underline{x} yx^2y^2x$$

by letting $x = uv$ and $y = vu$, for any $u, v \in \mathcal{P}_3$, cf. (1.10) in Construction 1.9.

Proof. \mathcal{P}_3 is isomorphic to the product $\mathfrak{A}_3^\times \times \diamond \mathfrak{A}_3^\times$ (Theorem 7.18), both \mathfrak{A}_3^\times and $\diamond \mathfrak{A}_3^\times$ are submonoids of $\text{TMat}_3(\mathbb{T})$, which by Theorem 3.2 satisfies the identity (3.6) for $n = 3$. \square

The maps \mathcal{C}_{mat} and $\mathcal{C}_{\text{mat}}^{\text{co}}$ By our construction, one sees that for any $w \in \mathcal{P}_n$ the characters (cf. (4.6) and (4.7))

$$\chi^+(\mathcal{C}_{\text{mat}}(\mathfrak{C}_w)) = \chi^\times(-\mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_w)), \quad \chi^\times(\mathcal{C}_{\text{mat}}(\mathfrak{C}_w)) = -\chi^\times(\mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_w)),$$

and thus, embedding $\wp_n(w)$ trivially in $2n \times 2n$ matrix C_w with $\mathcal{C}_{\text{mat}}(\mathfrak{C}_w)$ and $\mathcal{C}_{\text{mat}}^{\text{co}}(\mathfrak{C}_w)$ as its diagonal blocks, by (7.9) we have

$$\chi^+(\wp_n(w)) = \chi^+(\mathcal{C}_{\text{mat}}(\mathfrak{C}_w)), \quad \chi^\times(\wp_n(w)) = 0.$$

Recall that χ^+ provides the maximal occurrence of a letter in w , cf. §4. Then, $\{C_w \mid w \in \mathcal{A}_n^*\} \subset \text{TMat}_{2n}(\mathbb{T})$ is a multiplicative monoid whose members all have multiplicative trace $\text{mtr} = 0$ (Definition 3.4).

7.5. Standard Young tableaux and the symmetric group.

The special structure of standard tableaux STab_n , via the corresponding standard n -configuration tableaux (Lemma 6.24), allows their faithful realization in terms of the tropical representation of the cloactic monoid \mathcal{K}_n (Theorem 5.6). (We use the terminology “realization” as STab_n does not form a monoid, and our maps here are set-theoretical maps.)

Theorem 7.20 (Realization of standard tableaux). *Writing $\mathfrak{T}_w = \text{tab}(\vec{w})$ and $\mathfrak{T}'_w = \text{tab}(\vec{w})$ for standard tableaux of \vec{w} and \vec{w} , i.e., \vec{w} is a permutation of a_1, \dots, a_n , then the map*

$$\mathcal{S}_n : \text{STab}_n \longrightarrow \mathfrak{A}_n^\times \times \mathfrak{A}_n^\times, \quad \mathfrak{T}_w \longmapsto (\mathcal{S}_{\text{mat}}(\mathfrak{T}_w), \mathcal{S}_{\text{mat}}(\mathfrak{T}'_w)), \quad (7.10)$$

is injective.

Proof. Compose Proposition 6.22 with Theorem 6.25. \square

By the bijective correspondence of the symmetric group S_n to standard Young tableaux [44], we define the set-theoretic map

$$\mathcal{H}_n : S_n \longrightarrow \mathfrak{A}_n^\times \times \mathfrak{A}_n^\times, \quad \sigma \longmapsto \mathcal{S}_n(\mathfrak{T}_\sigma), \quad (7.11)$$

induced from (7.10), and thus obtain a tropical matrix realization of S_n .

A supplementary tropical view to S_n is briefly as follows. Let $s_i \in S_n$ be the simple transposition that permutes i with $i + 1$ and leaves the other elements of $\{1, \dots, n\}$ unchanged. The set $\{s_1, \dots, s_{n-1}\}$ of all these transpositions (called Coxeter transpositions) generates the symmetric group S_n , and thus every permutation $\sigma \in S_n$ can be written in terms of s_i ’s as a word $s_{i_1}s_{i_2} \cdots s_{i_m}$ with $i_1, \dots, i_m \in \{1, \dots, n-1\}$. To get a reduced form of σ we take $s_{i_1}s_{i_2} \cdots s_{i_m}$ with minimal m .

Consider the transposition s_i as a word over $\{1, \dots, n\}$. It consists of exactly n letters, each of which appears once. Applying the map (7.11) to transpositions s_i , we obtain the induced (set-theoretic) map

$$\mathcal{Q}_n : S_n \longrightarrow \mathfrak{A}_n^\times \times \mathfrak{A}_n^\times, \quad s_i \longmapsto \mathcal{S}_n(\mathfrak{T}_{s_i}), \quad (7.12)$$

determined now by the generators’ mapping. This realization has special properties, for example all $\mathcal{Q}_n(s_i)$ are quasi-idempotents, i.e., $\mathcal{Q}_n(s_i) = q(\mathcal{Q}_n(s_i))^2$ with a fixed $q \in \mathbb{R}$, establishing an important linkage to Hecke algebras [16].

8. REMARKS AND OPEN PROBLEMS

The first question arises from our results concerns a generalization of Theorem 7.18.

Problem 8.1. *Is there a faithful linear representation of the plactic monoid of rank n by triangular (or nonsingular) tropical matrices?*

By Theorem 3.2, a positive answer to this question solves the problem (cf. [27]):

Problem 8.2. *Does the plactic monoid of rank n satisfy a nontrivial semigroup identity?*

This paper shows that the plactic monoid \mathcal{P}_3 of rank 3 admits all the semigroup identities satisfied by the monoid $\text{TMat}_3(\mathbb{T})$ of 3×3 triangular tropical matrices (Corollary 7.19). This naturally leads to the converse question:

Problem 8.3. *Do \mathcal{P}_3 and $\text{TMat}_n(\mathbb{T})$ satisfy exactly the same identities?*

A few fundamental facts on the symmetric group S_n are well known:

- the irreducible representations of S_n are in canonical bijection with partitions λ of n ;
- the dimension of the irreducible representation corresponding to a given partition λ of n is exactly d_λ – the number of standard Young tableaux of shape λ .

Actually, these facts were the original motivation for studying Young tableaux, and many combinatorial results on Young tableaux can be expressed by using representations of S_n , e.g., see [44].

The bijective correspondence of elements of S_n to standard Young tableaux, which in their turn are faithfully realizable in $\mathfrak{A}_n^\times \times \mathfrak{A}_n^\times$ (Theorem 5.6), leads to a new approach for studying representations of S_n . Hook’s formula computes the numbers d_λ in combinatorial terms of Young tableaux, these numbers are possibly reflected in their tropical matrix realization.

Problem 8.4. *Can the d_λ ’s be extracted from tropical realizations in an algebraic way?*

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APPENDIX A. PROOFS AND COMPLETIONS TO §7.1

In this appendix we bring additional details for §7.1, in particular the full detailed proof of Theorem 7.2. We use the terminology in §1.2.

Example A.1. Let $\mathcal{P}_3 := \langle a, b, c \rangle$ be the plactic monoid of rank 3, for which the co-mirrors (1.4) of generators are

$$a' := {}^{\text{co}}\mathfrak{M}(a) = ba, \quad b' := {}^{\text{co}}\mathfrak{M}(b) = ca, \quad c' := {}^{\text{co}}\mathfrak{M}(c) = cb,$$

ordered as $a' < b' < c'$. Then a', b', c' admit the Knuth relations (KNT) and thus the congruence \equiv_{plc} of \mathcal{P}_3 implies the equivalence $\equiv_{\text{plc}}^{\text{co}}$. Indeed, for KN1 we have ⁵

$$a' c' b' = ba cb ca = c bb aab = cb ba ca = c' a' b', \quad \mathfrak{C} = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 0 & \\ \hline 2 & 0 & 1 \\ \hline \end{array}, \quad (\text{A.1})$$

and

$$a' b' a' = ba ca ba = c bb aaa = ca ba ba = b' a' a', \quad \mathfrak{C} = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 0 & \\ \hline 3 & 0 & 0 \\ \hline \end{array}, \quad (\text{A.2})$$

when $a' < b'$.

For the second relation KN2 we have ⁵

$$b' a' c' = ca ba cb = c bc aab = ca cb ba = b' c' a', \quad \mathfrak{C} = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 1 & 1 & \\ \hline 2 & 1 & 0 \\ \hline \end{array}, \quad (\text{A.3})$$

⁵The middle terms are presented in their canonical tableau form, that is as elements of the plactic monoid, as discussed in §6.1.

where

$$c' b' c' = cb ca cb = c bc abc = cb cb ca = c' c' b', \quad \mathfrak{C} = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 1 & 1 & \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \quad (\text{A.4})$$

for $b' < c'$.

For the two other 3-letter words we get

$$\begin{aligned} a' b' c' &= ba ca cb = bcc aab, & \mathfrak{C} &= \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 2 & 0 & \\ \hline 2 & 0 & 0 \\ \hline \end{array}, \\ c' b' a' &= cb ca ba = cc bb aa, & \mathfrak{C} &= \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 1 & 2 & \\ \hline 2 & 1 & 0 \\ \hline \end{array}. \end{aligned}$$

So we see that the Knuth relations (KNT) are “faithfully” preserved.

Let $a'_\ell := {}^{\text{co}}\mathfrak{M}(a_\ell)$ be the co-mirror of $a_\ell \in \mathcal{A}_n$, which is a word in \mathcal{A}_n^* , and set $\ell' = n - \ell + 1$. The n -configuration tableau of a'_ℓ is tableau

$$\text{ctab}(a'_\ell) = \ell' \begin{array}{|c|c|c|c|c|c|c|} \hline 0 & & & & & & \\ \hline 0 & 1 & & & & & \\ \hline \vdots & \vdots & \ddots & & & & \\ \hline 0 & 1 & 0 & & & & \\ \hline 1 & 0 & \cdots & & 0 & & \\ \hline \vdots & \vdots & \ddots & & & \ddots & \\ \hline 1 & 0 & \cdots & 0 & & \cdots & 0 \\ \hline \end{array} \quad (\text{A.5})$$

whose ℓ' -diagonal is zero; in the case that $\ell = n$ (i.e., when $\ell' = 1$) the left column is empty.

Theorem 7.2. *The co-mirrors (1.4) of the generators a_1, \dots, a_n of the plactic monoid \mathcal{P}_n , $a'_\ell := {}^{\text{co}}\mathfrak{M}(a_\ell)$, $\ell = 1, \dots, n$, ordered as $a'_1 < a'_2 < \dots < a'_n$, admit the Knuth relations (KNT) and thus the congruence \equiv_{plc} of \mathcal{P}_n implies the equivalence $\equiv_{\text{plc}}^{\text{co}}$.*

Proof of Theorem 7.2. Proof by induction on n . The case of $n = 3$ has been proven in Example A.1. Assuming the implication holds for $n - 1$, we prove it by cases for n , heavily basing on the Encoding Algorithm 6.19 of configuration tableaux.

For an easy exposition, we write the words a'_1, \dots, a'_n in \mathcal{A}_n^* , each is of length $(n - 1)$, as a table

$$\begin{array}{l} a'_1 : \\ a'_2 : \\ a'_3 : \\ \vdots \\ a'_{n-1} : \\ a'_n : \end{array} \begin{array}{|c|c|c|c|c|c|} \hline & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ \hline a_n & & a_{n-2} & \cdots & a_2 & a_1 \\ \hline a_n & a_{n-1} & & \ddots & & \vdots \\ \hline \vdots & & & \ddots & & \\ \hline a_n & \cdots & & a_3 & & a_1 \\ \hline a_n & \cdots & & a_3 & a_2 & \\ \hline \end{array}, \quad (\text{A.6})$$

where the empty spaces stand for absent letters. To indicate that an n -configuration tableau \mathfrak{C} is considered with respect to the sub-alphabet $\{a_k, \dots, a_n\} \subset \mathcal{A}_n$, we denote it by $\mathfrak{C}^{(k:n)}$; $\mathfrak{C}_w^{(n)} := \mathfrak{C}_w^{(1:n)}$ denotes the n -configuration tableau of a word $w \in \mathcal{A}_n^*$ over the whole alphabet $\mathcal{A}_n = \{a_1, \dots, a_n\}$.

As the word a'_ℓ has a decreasing order, the encoding of its i 'th letter in $\mathfrak{C}^{(n)}$ “bumps” all the pervious letters, resulting in a tableau of the form (A.5), which is similar to encoding its $(i - 1)$ -prefix in a configuration tableau of a fewer letters. Thus, we obtain the following observations for the encoding $a'_\ell \mapsto \mathfrak{C}$ of a'_ℓ in \mathfrak{C} .

(A) If $\ell < n$ then $a'_\ell = ua_1$ and the upper part of $\mathfrak{C}_{ua_1}^{(n)}$ is equal to $\mathfrak{C}_u^{(2:n)}$.

(B) When $\ell = n$, the right part of $\mathfrak{C}_{a'_n}^{(n)}$ is equal to $\mathfrak{C}_{a'_n}^{(2:n)}$.

Let $a' = a'_p, b' = a'_q, c' = a'_r$, where $p \leq q \leq r$. By (KNT) we have to prove the following equalities

KN1: $a' c' b' = c' a' b'$ if $p \leq q < r$,

KN2: $b' a' c' = b' c' a'$ if $p < q \leq r$.

$$\text{KN1 } (a c b = c a b) : \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 2 & 0 & 1 & \dots & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 2 & 0 & 1 & \dots & 0 \\ \hline \end{array} .$$

(In this case, KN2 is not relevant here.)

- (d) $p < n - 1 < q = r = n$: Let $a'_p = ua_2a_1$ and $a'_q = a'_r = va_3a_2$. The first row of $\mathfrak{C}_{a'_q a'_p}$ is $110 \cdots 0$, then $a'_r \rightarrow \mathfrak{C}_{a'_q a'_p}$ increments $\lambda_{1,2}$ to obtain $\mathfrak{C}_{a'_q a'_p a'_r}|_{\text{row}_1} = 120 \cdots 0$. On the other hand $\mathfrak{C}_{a'_q a'_r}|_{\text{row}_1} = 020 \cdots 0$, and $a'_p \rightarrow \mathfrak{C}_{a'_q a'_r}$ gives $\mathfrak{C}_{a'_q a'_r a'_p}|_{\text{row}_1} = 120 \cdots 0$. Thus

$$\text{KN2 } (b a c = b c a) : \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 1 & 2 & 0 & \dots & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 1 & 2 & 0 & \dots & 0 \\ \hline \end{array} .$$

by induction on $a'' = ua_2$, $b'' = c'' = va_3$. (In this case, KN1 is not relevant here.)

- (e) $p = n - 1 < q = r = n$: Write $a'_p = ua_3a_1$ and $a'_q = a'_r = va_3a_2$ (e.g. see (A.4)). The first row of $\mathfrak{C}_{a'_q a'_p}$ is $1010 \cdots 0$, then $v \rightarrow \mathfrak{C}_{a'_q a'_p}$ fills the forth column by 1, which is bumped by $a_3 \rightarrow \mathfrak{C}_{a'_q a'_p v}$. So $\mathfrak{C}_{a'_q a'_p va_3}|_{\text{row}_1} = 1020 \cdots 0$, and $a_2 \rightarrow \mathfrak{C}_{a'_q a'_p va_3}$ gives $\mathfrak{C}_{a'_q a'_p a'_r}|_{\text{row}_1} = 1110 \cdots 0$, as $\lambda_{1,3}$ is decremented by encoding a_2 . On the other hand $\mathfrak{C}_{a'_q a'_r}|_{\text{row}_1} = 020 \cdots 0$, and $a'_p \rightarrow \mathfrak{C}_{a'_q a'_r}$ gives $\mathfrak{C}_{a'_q a'_r a'_p}|_{\text{row}_1} = 1110 \cdots 0$. Thus

$$\text{KN2 } (b a c = b c a) : \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 1 & 1 & 1 & \dots & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline 1 & 1 & 1 & \dots & 0 \\ \hline \end{array} ,$$

since $u = v$. (In this case, KN1 is not relevant here.)

The above cases show that involving a new letter reflects inductively on extending the tableaux by a bottom row, and the proof is completed. \square